

# Méthodes Mathématiques pour la Physique

## Cours: Physique Master 1 (ENS)

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*In order of appearance:* Sir Isaac Newton (1643–1727); Joseph Louis Lagrange (1736–1813); Pierre-Simon Laplace (1749–1827); Adrien-Marie Legendre (1752–1833); Jean Baptiste Joseph Fourier (1768–1830); Johann Carl Friedrich Gauss (1777–1855); Siméon Denis Poisson (1781–1840); Carl Gustav Jacob Jacobi (1804–1851); William Rowan Hamilton (1805–1865); Georg Friedrich Bernhard Riemann (1826–1866); James Clerk Maxwell (1831–1879); Felix Christian Klein (1849–1925); Jules Henri Poincaré (1854–1912); David Hilbert (1862–1943); Hermann Minkowski (1864–1909); Elie Joseph Cartan (1869–1951).

*Two truths can never contradict themselves.*

(Galileo Galilei, 'Il saggiatore' (The Assayer) 1623)

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Lecture 4: incl. 2.4.2. Lecture 5: incl. 3.2.3. Lectures 6,7: rest.

## Selected Books

A. Alastuey, M. Magro, P. Pujol: *Physique et outils mathématiques: méthodes et exemples*, EDP Sciences & CNRS Editions (2008).

W. Appel: *Mathématiques pour la Physique et les Physiciens*, 3e édition, H & K Editions (2006).

G.B. Arfken, H.-J. Weber: *Mathematical Methods for Physicists*, 5th edition, Academic Press (2001).

V.I. Arnol'd: *Mathematical Methods of Classical Mechanics*, Springer 2<sup>nd</sup> edition (1989).

T.L. Chow: *Mathematical Methods for Physicists: a concise introduction*, Cambridge Univ. Press (2003).

R. Courant, D. Hilbert: *Methods of Mathematical Physics*, Wiley (1989).

L. Debnath: *Nonlinear Partial Differential Equations for Scientists and Engineers*, 2nd edition, Birkhäuser (2004).

L.P. Eisenhart: *An Introduction to Differential Geometry*, Princeton Univ. Press, (1949).

H. Flanders: *Differential Forms with Applications to the Physical Sciences*, Academic Press (1963).

J. Hladík: *Le calcul tensoriel en physique*, Masson (1993).

B. Schutz: *Geometrical Methods of Mathematical Physics*, Cambridge Univ. Press (1980).

M.A. Spivak: *Comprehensive Introduction to Differential Geometry*, 5 Vols., Publish or Perish Inc. (1970).