Formation Sciences de la Matière
Cours : Physique Master 1 (ENS)

Méthodes Mathématiques pour la Physique

TD 1

Thomas Buchert, CRAL, Observatoire de Lyon thomas.buchert@ens–lyon.fr
Rasha Boulos, ENS rasha.boulos@ens–lyon.fr
Clément Tauber, ENS clement.tauber@ens–lyon.fr

1.1. The Euler–Lagrange equation in classical mechanics

By considering the increment of the Lagrangian, $\mathcal{L}(\mathbf{x}+\delta\mathbf{x},\dot{\mathbf{x}}+\delta\dot{\mathbf{x}})-\mathcal{L}(\mathbf{x},\dot{\mathbf{x}})$, we saw that the differential of the functional $\mathcal{L}$ defines its variation, similar as the differential of a function defines its derivative. The integral of the increment defines the variation of the action :

$$\delta_\varepsilon \mathcal{S} := \int_{\tau_1}^{\tau_2} \mathcal{L}(\mathbf{x}+\delta\mathbf{x},\dot{\mathbf{x}}+\delta\dot{\mathbf{x}})d\tau - \int_{\tau_1}^{\tau_2} \mathcal{L}(\mathbf{x},\dot{\mathbf{x}})d\tau \quad . \tag{1.1}$$

Introduce a displacement vector field $\mathbf{z}(\mathbf{x})$ and realize the virtual displacements of a particle path as follows :

$$\delta\mathbf{x}(\tau) = \varepsilon\mathbf{z}(\tau) \quad ; \quad \delta\dot{\mathbf{x}}(\tau) = \varepsilon\dot{\mathbf{z}}(\tau) = \frac{d}{d\tau}\delta\mathbf{x}(\tau) \quad . \tag{1.2}$$

Compute $\delta_\varepsilon \mathcal{S}$ from its definition (1.1) and derive, under the stationarity condition $\delta_\varepsilon \mathcal{S} = 0$, the Euler–Lagrange equation determining the paths of stationary length :

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{\partial \mathcal{L}}{\partial x^i} = 0 \quad ; \quad i = 1,2,3 \quad . \tag{1.3}$$

Consider now the following generalized action principle :

$$\delta_\varepsilon \mathcal{S} := \delta \int_{\tau_1}^{\tau_2} \left(\mathcal{L} + \sum_{k=1}^{p} \lambda_k(\tau) g_k(\mathbf{x},\tau) \right) d\tau \ = \ 0 \ , \tag{1.4}$$

where we impose the p relations $g_k(\mathbf{x},\tau) = 0$, i.e. the functions g_k define hypersurfaces in space. We wish to find the solution with the presence of such *forcing conditions*. Perform the variation of the generalized action principle and explain, how we can compute the *Lagrangian multipliers* $\lambda_k(\tau)$.

1.2. Geodesics in Euclidean space and the Galilei–transformation

Consider Euclidean three–space $\mathbb{E}^3$ endowed with the cartesian coordinates x^1, x^2, x^3, and the Euclidean line element,

$$ds^2 = \delta_{ij} dx^i dx^j \quad ; \quad \Rightarrow \quad ds = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2} \ . \tag{1.5}$$

Now, introduce a curve $\mathcal{C} \subset \mathbb{E}^3$ parametrized by τ, i.e. $dx^i = \dot{x}_i d\tau$ to rewrite the line element as $ds = |\dot{\mathbf{x}}| d\tau = \mathcal{L} d\tau$, where in the last equation we have defined the Lagrangian. Compute the canonical momenta p_i, and find the general solution of the Euler–Lagrange equations by employing the invariant parameter s. Find the solutions by employing the Newtonian time t as the curve parameter instead of s. Discuss the interpretation of the simplest of these solutions as a *Galilei–transformation*.

1.3. The Klein–Gordon equation

Consider a scalar field $\Phi(\mathbf{x}^\mu)$ on *Minkowski spacetime*, which is a four–dimensional Euclidean space endowed with the cartesian coordinates $(x^0 := ict, x^1, x^2, x^3)$.
Introduce the Lorentz–covariant Lagrangian $\mathcal{L}(\partial^\mu \Phi, \Phi, t) := a^2(t) \left(\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - V(\Phi) \right)$ with kinetic and potential terms and an explicit function of the time–parameter $\tau \equiv t$. Show that the corresponding least action principle yields the relativistic wave equation (an overdot denotes partial time–derivative) :

$$\Box \Phi := -\frac{1}{c^2} \ddot{\Phi} + \Delta \Phi \quad ; \quad \Box \Phi - \frac{2}{c^2} \frac{\dot{a}}{a} \dot{\Phi} + \frac{\partial V(\Phi)}{\partial \Phi} = 0 \ . \tag{1.6}$$

Discuss the role of the function $a(t)$.

In what follows we set $a \equiv 1$.

Recall that a quantum mechanical generalization would require the plane wave solutions $\Phi(\mathbf{x}, t) = \exp(i\mathbf{kx} - i\omega t)$ to satisfy the de Broglie and Einstein relations $\mathbf{p} = \hbar \mathbf{k}$ and $E = \hbar \omega$. Recalling the relativistic energy momentum equation $E^2 = E_0^2 + \mathbf{p}^2 c^2$, with the energy equivalent $E_0 = m_0 c^2$ of the restmass m_0, derive the relativistic *Klein–Gordon equation* from these relations :

$$\Box \Phi - \frac{m_0^2 c^2}{\hbar^2} \Phi = 0 \ . \tag{1.7}$$

Discuss the result in terms of the Lagrangian of the quantum mechanical relativistic system. What is the interpretation of the scalar field Φ ?

1.4. Geodesics on the unit sphere

Consider a sphere embedded into $\mathbb{E}^3$, and write the Euclidean line element in spherical coordinates, $ds = \sqrt{d\vartheta^2 + sin^2\vartheta d\varphi^2}$. Calculate the canonical momenta p_ϑ and p_φ from Hamilton's function $\mathcal{H}$ inferred from this metric. Then, set up the Hamilton–Jacobi differential equation for the action $\mathcal{S}$ and solve it in order to compute the geodesics on the spherical surface.