Formation Sciences de la Matière
Cours : Physique Master 1 (ENS)

Méthodes Mathématiques pour la Physique

TD 2

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2.1. Light deflection in the Schwarzschild metric

Consider a special solution of Einstein's theory of general relativity providing a specific form of a space-time metric modelling the gravitational field. We only need to know that light propagation follows geodesics in this four–dimensional metric (${}^{(4)}ds^2 = 0$, with the line–element ${}^{(4)}ds$ in spacetime), then we can apply the eikonal equation. We give this metric in the form of the *Schwarzschild line element* in spherical coordinates $(\tilde{r}, \vartheta, \varphi)$,

$$ {}^{(4)}ds^2 = -\left(1 - \frac{2\mu}{\tilde{r}}\right) c^2 dt^2 + \left(1 - \frac{2\mu}{\tilde{r}}\right)^{-1} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2 \;\; ; \;\; d\Omega^2 := d\vartheta^2 + \sin^2(\vartheta) d\varphi^2 \;\; , \tag{2.1} $$

where $\mu := GM/c^2 =: R_S/2$, with R_S denoting the so–called *Schwarzschild–Radius* that describes a critical radius for the evolution into a *Black Hole*. (For an object with the mass of our Sun we have $R_S \approx 3$ km). In this metric we can consider $2\mu/\tilde{r} = -\Phi(\tilde{r})/c^2$ as a radial gravitational potential.
Since the gravitational field described by this metric is only defined up to diffeomorphisms (coordinate transformations) in Einstein's theory, we can rewrite the above metric form, that was originally proposed by Schwarzschild himself, by introducing a new radial coordinate r.
Determine first $\chi(r)$ and $\psi(r)$ such that the line element (2.1) can be cast into the following form, that turns out to be more convenient for our problem :

$$ {}^{(4)}ds^2 = -\chi^2(r) c^2 dt^2 + \frac{1}{\psi^2(r)} |d\mathbf{r}|^2 \;\; ; \;\; \mathbf{r} = (r, \vartheta, \varphi) \;\; . \tag{2.2} $$

Then, discuss why the function $n(r) := c/|d\mathbf{r}/dt| = 1/(\chi(r)\psi(r))$ can be considered a spatial diffraction index, and give its explicit form in terms of the coordinates r. Consider further the regime $r \gg R_S$, and Taylor expand the function $n^2(r)/2$. Set up the eikonal equation for the linearized problem and demonstrate that the resulting equation of motion is analoguous to the *Kepler problem* for the potential $U = -n^2/2$:

$$ \ddot{\mathbf{r}} = -\boldsymbol{\nabla} U \quad \text{with} \quad U := -\frac{2\mu}{r} = \frac{\Phi}{c^2} \;\; . \tag{2.3} $$

Employ now the analogy to the Kepler problem by defining the energy $E := 1/2m|\dot{\mathbf{r}}^2| + U(r)$, with $m = 1$ here, and determine the value of E from the eikonal equation. Calculate, for the numerical example of the Sun ($R_\odot \approx 7 \cdot 10^5$ km ; $M_\odot \approx 2 \cdot 10^{30}$ kg), the deflection angle β with which a light ray is deflected in the gravitational field of the Sun (draw a picture to define this angle).

2.2. The geodesic equations in a Riemannian space

Extremalize, between two points P and Q, the Riemannian line segment (a space curve $x^i = x^i(\tau)$) :

$$\mathcal{S} = \int_P^Q ds = \int_P^Q \sqrt{g_{ij} dx^i dx^j} = \int_{\tau_1}^{\tau_2} \sqrt{g_{ij} \frac{dx^i}{d\tau}\frac{dx^j}{d\tau}} d\tau =: \int_{\tau_1}^{\tau_2} \mathcal{L} d\tau \quad , \tag{2.4}$$

using the Euler–Lagrange equations. Note that the coefficients g_{ij} of the Riemannian metric depend on $\mathbf{x}$ and are symmetric. Then, derive the following set of geodesic equations :

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0 \quad ; \quad \Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell}\left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right) \quad , \tag{2.5}$$

with the *Christoffel symbols* $\Gamma^i{}_{jk}$ of a so–called *symmetric spatial connection*, and the inverse metric tensor $g^{i\ell}$ with $g^{i\ell} g_{\ell k} = \delta^i{}_k$. Discuss the final result for the case of an Euclidean metric.
Hint : Make use of the symmetry with respect to j and k in the following expression in order to obtain the form suggested by Christoffel :

$$\frac{\partial g_{\ell j}}{\partial x^k}\frac{dx^k}{d\tau}\frac{dx^j}{d\tau} = \frac{1}{2}\left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j}\right)\frac{dx^k}{d\tau}\frac{dx^j}{d\tau} \quad . \tag{2.6}$$

Note also that the metric depends on $\mathbf{x}(\tau)$, i.e. also implicitly on the parameter τ. In the expressions obtained you may change to the arc length s as parametrization of the geodesics, i.e. you can replace the square root of the line–element by $ds/d\tau$ during the calculation.

2.3. The transformation of field equations

Assume a coordinate transformation $\mathbf{y} \mapsto \mathbf{x} = \mathbf{f}(\mathbf{y}, t)$, which may depend on time, as given. Introduce functional determinants to write the *electric field strength gradient* subjected to the coordinate transformation $\mathbf{f}$ as follows :

$$E^i{}_{,j} = \frac{1}{2J}\epsilon_{jpq}\mathcal{J}(E^i, f^p, f^q) \quad . \tag{2.7}$$

Calculate the curl and the divergence of the field strength gradient in order to write the transformed field equations of electrostatics for the electric field strength $\mathbf{E}(\mathbf{x}, t)$:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\varrho_e \quad ; \quad \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0} \quad . \tag{2.8}$$

Try to eliminate the Levi–Civita symbols from these expressions.

Hint : Make yourself first familiar with the identitites, $\varepsilon_{ijk}\varepsilon^{ipq} = \delta_j{}^p\delta_k{}^q - \delta_j{}^q\delta_k{}^p$ and $E_{i,j} = \delta_{\ell i} E^\ell{}_{,j}$, by calculating from the components of the curl of $\mathbf{E}$ in the form $(\boldsymbol{\nabla} \times \mathbf{E})^i = \epsilon^{ijk} E_{k,j}$ the relation $\epsilon_{ijk}(\boldsymbol{\nabla} \times \mathbf{E})^k = -2E_{[i,j]}$ with $E_{[i,j]} := 1/2(E_{ij} - E_{ji})$.