# Formation Sciences de la Matière
# Cours : Physique Master 1 (ENS)

*Méthodes Mathématiques pour la Physique*

**TD 3**


Thomas Buchert, CRAL, Observatoire de Lyon — thomas.buchert@ens–lyon.fr
Rasha Boulos, ENS — rasha.boulos@ens–lyon.fr
Clément Tauber, ENS — clement.tauber@ens–lyon.fr


## 3.1. Green's function for a point charge

The electric potential $\Phi(\mathbf{x})$ at a point $\mathbf{x}$ due to a point charge $q_e$ located at a point $\tilde{\mathbf{x}}$ is given by

$$\Phi(\mathbf{x}) = \frac{q_e}{|\mathbf{x}-\tilde{\mathbf{x}}|} \quad , \tag{3.1}$$

and must obey Poisson's equation $\Delta\Phi = 4\pi\varrho_e$ with

$$\Phi(\mathbf{x}) = 4\pi \int_{\tilde{\mathcal{D}}} \mathfrak{G}(\tilde{\mathbf{x}},\mathbf{x})\varrho_e(\tilde{\mathbf{x}})\, d\tilde{\mathbf{x}} \quad . \tag{3.2}$$

Show that Green's function, obeying the operator equation $\Delta\mathfrak{G}(\mathbf{x},\tilde{\mathbf{x}}) = \delta(\mathbf{x}-\tilde{\mathbf{x}})$, is determined to be

$$\mathfrak{G}(\mathbf{x},\tilde{\mathbf{x}}) = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\tilde{\mathbf{x}}|} \quad . \tag{3.3}$$

*Hint :* Employ Fourier's transformation of Green's function, obeying $|\mathbf{k}|^2\hat{\mathfrak{G}}(\mathbf{k}) = 1$, and calculate the Fourier integral using spherical coordinates in Fourier space with the $z$–axis pointing into the direction of $(\mathbf{x}-\tilde{\mathbf{x}})$. Note : $\int_0^\infty \frac{\sin a\xi}{\xi} d\xi = \pi/2$ for $a > 0$ and $-\pi/2$ for $a < 0$.

## 3.2. General solution of the wave equation

First, let us recall the wave equations derived from Maxwell's theory. Maxwell's equations for the electric and magnetic field strengths $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ read (we use Gaussian units ; for a clear presentation of classical electrodynamics see Jackson's book) :

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{E} &= -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} \quad ; \qquad & \boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi\varrho_e \quad ; \\ \boldsymbol{\nabla}\times\mathbf{B} &= \frac{4\pi}{c}\mathbf{j}_e + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} \quad ; \qquad & \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \quad , \end{aligned} \tag{3.4}$$

where $\varrho_e$ is the charge density field, and $\mathbf{j}_e := \varrho_e \mathbf{v}$ the charge current density with the associated flow velocity $\mathbf{v}$. The equations for $\mathbf{E}$ are *Faraday's law* and *Coulomb's law*, the equations for $\mathbf{B}$ are *Ampères law* and the law of absence of magnetic monopoles. Recall now that with

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E} = -\tfrac{1}{c}\tfrac{\partial}{\partial t}\nabla \times \mathbf{B} = -\tfrac{4\pi}{c^2}\tfrac{\partial}{\partial t}\mathbf{j}_e - \tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}\mathbf{E} \ ;$$
$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \Delta \mathbf{B} = \tfrac{4\pi}{c}\nabla \times \mathbf{j}_e - \tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}\mathbf{B} \ , \tag{3.5}$$

we obtain the following wave equations :

$$\Box \mathbf{E} \ = \ 4\pi \nabla \varrho_e + \frac{4\pi}{c^2}\frac{\partial}{\partial t}\mathbf{j}_e \quad ; \quad \Box \mathbf{B} \ = \ -\frac{4\pi}{c}\nabla \times \mathbf{j}_e \ , \tag{3.6}$$

with the d'Alembertian $\Box := \partial^2/\partial x_0^2 + \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$, written in Minkowski spacetime with coordinates $(x_0 = ict, x_1, x_2, x_3)$.

For vanishing sources, i.e., $\varrho_e = 0$ and therefore $\mathbf{j}_e = \mathbf{0}$, these equations describe the propagation of electromagnetic waves at speed $c$.

Now, try to solve the wave equation for the electric field potential $\Phi(\mathbf{x}, t)$, $\mathbf{E} = \nabla \Phi(\mathbf{x}, t)$, by determining its Green's function for an arbitrary time–dependent and space–continuous charge distribution, but restricted to non–moving charges, $\mathbf{v} = \mathbf{0}$. Hence, we consider the following wave equation :

$$\mathfrak{O}\Phi(\mathbf{x}, t) = 4\pi \varrho_e(\mathbf{x}, t) \quad ; \quad \mathfrak{O} = -c^{-2}\partial_t^2 + \Delta \ . \tag{3.7}$$

Show that the potential is given by :

$$\Phi(\mathbf{x}, t) = 4\pi \int \mathfrak{G}_R(\mathbf{x} - \tilde{\mathbf{x}}, t - \tilde{t})\varrho_e(\tilde{\mathbf{x}}, \tilde{t}) \, d\tilde{\mathbf{x}} \, d\tilde{t} = \int \frac{\varrho_e(\tilde{\mathbf{x}}, t - \frac{1}{c}|\mathbf{x} - \tilde{\mathbf{x}}|)}{|\mathbf{x} - \tilde{\mathbf{x}}|} \, d\tilde{\mathbf{x}} \ . \tag{3.8}$$

## 3.3. Transformation of a model equation

Consider the following model equation for a vector field $\mathbf{v}(\mathbf{x}, t)$ that may arise in a specific physical context for a continuous medium with density $\varrho(\mathbf{x}, t)$ :

$$\frac{d}{dt}\mathbf{v} - h\mathbf{v} = \frac{h}{4\pi G \varrho}\frac{\partial \beta}{\partial \varrho}\Delta \mathbf{v} \ , \tag{3.9}$$

where $h = h(t)$, $\beta = \beta(\varrho)$, $G = const.$, and $d/dt$ is the Lagrangian time–derivative.

Show that the following time–dependent transformation of $\mathbf{v}$ and the change of the time–variable, $t \to \tau$,

$$\mathbf{v} =: \alpha(t)\boldsymbol{\nu} \ ; \ \frac{\dot{\alpha}}{\alpha} = h(t) \ ; \ \dot{\tau} = \alpha \ , \tag{3.10}$$

leads to the model equation :

$$\frac{d}{d\tau}\boldsymbol{\nu} = \frac{\partial}{\partial \tau}\boldsymbol{\nu} + \boldsymbol{\nu} \cdot \nabla \boldsymbol{\nu} = \mu(\varrho, t)\Delta \boldsymbol{\nu} \quad \text{with} \quad \mu := \frac{1}{\varrho}\frac{\partial \beta}{\partial \varrho}\frac{h(t)}{4\pi G \alpha(t)} \ . \tag{3.11}$$

For $\mu = \mathrm{const.}$ this equation is known as the *three–dimensional Burgers equation*. In general, the coefficient in front of the Laplacian is density– and time–dependent. Notwithstanding, *Burgers' equation* is still a good model in many circumstances.