Formation Sciences de la Matière
Cours : Physique Master 1 (ENS)

Méthodes Mathématiques pour la Physique

TD 4

Thomas Buchert, CRAL, Observatoire de Lyon thomas.buchert@ens–lyon.fr
Rasha Boulos, ENS rasha.boulos@ens–lyon.fr
Clément Tauber, ENS clement.tauber@ens–lyon.fr

4.1. Exact solution of the linear diffusion equation

First, recall that by applying Fourier's transformation to the linear diffusion equation for the potential $U(\mathbf{x}, \tau)$,

$$\frac{\partial U}{\partial \tau} = \mu \Delta U \quad , \tag{4.1}$$

we obtain the following solution in Fourier space :

$$\hat{U}(\mathbf{k}, \tau) = \hat{U}_0(\mathbf{k}) e^{-\mu|\mathbf{k}|^2 \tau} \quad ; \quad \tau(t_0) = 0 \quad . \tag{4.2}$$

Transform, in the next step, this expression back to Eulerian space by the inverse Fourier transformation to arrive at the following representation of the solution of the linear diffusion equation :

$$U(\mathbf{x}, \tau) = \left(\frac{1}{4\pi\mu\tau} \right)^{\frac{3}{2}} \int d^3y \; U_0(\mathbf{y}) e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{4\mu\tau}} \quad . \tag{4.3}$$

Hint : Use the Fourier transform of the initial potential,

$$\hat{U}_0(\mathbf{k}) = \int d^3y \, U_0(\mathbf{y}) e^{i\mathbf{k}\cdot\mathbf{y}} \quad , \tag{4.4}$$

and employ the inverse Fourier representation of a *Gaussian distribution* :

$$e^{-\lambda \mathbf{y}^2} = \frac{1}{(2\pi)^3} \left(\frac{\pi}{\lambda} \right)^{3/2} \int d^3k \, e^{-i\mathbf{k}\cdot\mathbf{y}} e^{-\frac{|\mathbf{k}|^2}{4\lambda}} \quad , \quad \lambda \in \mathbb{R} \quad , \quad \lambda > 0 \quad , \tag{4.5}$$

for an Eulerian variable $\mathbf{y}$. If you are using another way for the proof, then you may need the Fourier representation of a *Dirac distribution*,

$$\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^3}\int d^3k\, e^{-i\mathbf{k}\dot{(}\mathbf{x}-\mathbf{y})} \quad , \tag{4.6}$$

and/or the inverse Fourier transform of the initial potential,

$$U_0(\mathbf{y}) = \frac{1}{(2\pi)^3}\int d^3k\, \hat{U}_0(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{y}} \quad . \tag{4.7}$$

To confirm this solution, show by explicit calculation that the representation for U in terms of a convolution with a Gaussian, Eq. (4.3), solves the linear diffusion equation (4.1).

4.2. Exact solution of Burgers' equation

Show, by transforming back to $\boldsymbol{\nu} = -2\mu\boldsymbol{\nabla}U/U$, that the solution of *Burgers' equation*

$$\frac{d}{d\tau}\boldsymbol{\nu} = \frac{\partial}{\partial\tau}\boldsymbol{\nu} + \boldsymbol{\nu}\cdot\boldsymbol{\nabla}\boldsymbol{\nu} = \mu\Delta\boldsymbol{\nu} \;\; ; \;\; \mu = const. \tag{4.8}$$

reads :

$$\boldsymbol{\nu} = \frac{\int_{\mathbb{R}^3}\frac{\mathbf{x}-\mathbf{X}}{\tau}e^{-\frac{1}{2\mu}\mathcal{G}(\mathbf{x},\mathbf{X},\tau)}\;d^3X}{\int_{\mathbb{R}^3}e^{-\frac{1}{2\mu}\mathcal{G}(\mathbf{x},\mathbf{X},\tau)}\;d^3X} \quad , \tag{4.9}$$

where the *generating function* (*cf. Chapter 1*),

$$\mathcal{G}(\mathbf{x},\mathbf{X},\tau) := S_0(\mathbf{X}) + \frac{(\mathbf{x}-\mathbf{X})^2}{2\tau} \quad , \tag{4.10}$$

solves the equation

$$\frac{\partial}{\partial\tau}\mathcal{G} + \frac{1}{2}(\boldsymbol{\nabla}G)^2 = 0 \quad \text{for fixed } \mathbf{X} \quad . \tag{4.11}$$

4.3. Examples of inverse Laplace transforms

Find the inverse Laplace transform $L^{-1}[F(p)] = f(x)$ of the following functions :

$$1) \quad L^{-1}\left[\frac{5}{p+2}\right] \quad ; \quad 2) \quad L^{-1}\left[\frac{1}{p^s}\right] \quad ; \quad s>0 \; . \tag{4.12}$$

Hint : Recall

$$L[e^{ax}] = \frac{1}{p-a} \quad ; \tag{4.13}$$

$$L[x^k] = \int_0^\infty e^{-px}x^k dx = \frac{1}{p^{k+1}}\int_0^\infty u^k e^{-u}du = \frac{\Gamma(k+1)}{p^{k+1}} \quad . \tag{4.14}$$