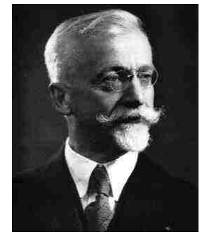
Formation Sciences de la Matière
Cours : Physique Master 1 (ENS)



Méthodes Mathématiques pour la Physique

TD 5

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5.1. Coordinate transformations and differential forms

As an example of a coordinate transformation we use the transformation from Lagrangian ($\mathbf{X}$) to Eulerian ($\mathbf{x}$) coordinates employed in continuum mechanics. Thus, we consider the position vector field $\mathbf{x} = \mathbf{f}(\mathbf{X}, t)$ as a coordinate transformation for a given parameter value t. Show that the *Lagrangian volume element* $\boldsymbol{d}^3 X := \boldsymbol{d}^3 x(t_0)$ is related to the *Eulerian volume element* $\boldsymbol{d}^3 x$ by :

$$\boldsymbol{d}^3 x := \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 \;=\; \boldsymbol{d}f^1 \wedge \boldsymbol{d}f^2 \wedge \boldsymbol{d}f^3 \;=\; J\, \boldsymbol{d}^3 X \;\;, \tag{5.1}$$

with the Jacobian $J(\mathbf{X}, t) = \det(\partial f^i / \partial X^k)$ of the transformation.

Hint : Recall the results on coordinate transformations of Chapter 1, and start with the expression for the Jacobian in the form

$$J \;=\; \mathcal{J}(f^1, f^2, f^3) \;=\; \frac{1}{6}\, \epsilon_{ijk} \epsilon^{\ell mn}\, f^i_{\;|\ell} f^j_{\;|m} f^k_{\;|n} \;\;, \tag{5.2}$$

where in the first expression we used a functional determinant $\mathcal{J}$, and where a vertical slash denotes partial derivative with respect to the Lagrangian coordinates. There are a number of ways to prove this and you may need the following identities :

$$\epsilon_{ijk}\epsilon^{ijk} = 6 \qquad ; \qquad dX^1 \wedge dX^2 \wedge dX^3 = \frac{1}{2}\epsilon_{ijk} dX^i \wedge dX^j \wedge dX^k \;\;,$$

and $\qquad \epsilon_{pqr}\epsilon^{\ell mn} \;=$

$$\delta_p^{\;\ell}\delta_q^{\;m}\delta_r^{\;n} \;+\; \delta_p^{\;m}\delta_q^{\;n}\delta_r^{\;\ell} \;+\; \delta_p^{\;n}\delta_q^{\;\ell}\delta_r^{\;m} \;-\; \delta_p^{\;m}\delta_q^{\;\ell}\delta_r^{\;n} \;-\; \delta_p^{\;n}\delta_q^{\;m}\delta_r^{\;\ell} \;-\; \delta_p^{\;\ell}\delta_q^{\;n}\delta_r^{\;m} \;\;.$$

5.2. Curl and divergence operators in different bases

We learned that, by transforming to Lagrangian coordinates, $\mathbf{x} = \mathbf{f}(\mathbf{X}, t)$, the acceleration field $\mathbf{b}$ can be expressed through the trajectory field by $\mathbf{b} = \ddot{\mathbf{f}}(\mathbf{X}, \mathbf{t})$.

Look now at the acceleration field as a function of Eulerian coordinates, and construct the corresponding *acceleration one–form*. Show that we obtain for the *Eulerian* curl of the acceleration form field and the *Eulerian* divergence of the acceleration vector field, expressed with the help of form operations, the following expressions *as functions of Lagrangian coordinates* :

$$\boldsymbol{d}\tilde{\mathbf{b}} = \boldsymbol{d}(\, \ddot{f}_i \, \boldsymbol{d}f^i \,) \; = \; \boldsymbol{d}\ddot{f}_i \wedge \boldsymbol{d}f^i \; = \; \delta_{ij}\boldsymbol{d}\ddot{f}^i \wedge \boldsymbol{d}f^j \;\; ; \tag{5.3}$$

$$\boldsymbol{d} \star \mathbf{b} = \boldsymbol{d}\left(\star(\delta_{ij}\ddot{f}^i \boldsymbol{d}f^j) \right) = 3\boldsymbol{d}\ddot{f}^{[1} \wedge \boldsymbol{d}f^2 \wedge \boldsymbol{d}f^{3]} \;\; . \tag{5.4}$$

Demonstrate further that the following *cubic* expressions for the curl and the divergence of $\mathbf{b}$, obtained in Chapter 1, have corresponding three–forms. For the curl equation this is in contrast to the two–form above (which is an example of how differential forms can simplify expressions) :

$$(\boldsymbol{\nabla} \times \mathbf{b})^i = \epsilon^{ijk}\ddot{f}_{k,j} = \frac{1}{J}\delta_{\ell k}\mathcal{J}(\ddot{f}^\ell, f^k, f^i) = \frac{1}{J}\delta_{\ell k}\epsilon^{pqr}\ddot{f}^\ell{}_{|p} f^k{}_{|q} f^i{}_{|r} \;\; . \tag{5.5}$$

$$(\boldsymbol{\nabla} \cdot \mathbf{b}) = b^k{}_{,k} = \frac{1}{2J}\epsilon_{\ell mn}\, \mathcal{J}(\ddot{f}^\ell, f^m, f^n) = \frac{1}{2J}\epsilon_{\ell mn}\epsilon^{pqr}\ddot{f}^\ell{}_{|p}, f^m{}_{|q}, f^n{}_{|r} \; . \tag{5.6}$$

Consider as an example the field equations $\boldsymbol{\nabla} \times \mathbf{b} = \mathbf{0}$. Write these equations in vector components corresponding to the two–form and three–form expressions for the curl of $\mathbf{b}$ above. With this example we show the above–mentioned simplification in explicit terms.
Hint : Note that

$$\begin{aligned} \frac{1}{2}\epsilon_{ijk}\, \boldsymbol{d}\ddot{f}^i \wedge \boldsymbol{d}f^j \wedge \boldsymbol{d}f^k \; &= \; \boldsymbol{d}\ddot{f}^1 \wedge \boldsymbol{d}f^2 \wedge \boldsymbol{d}f^3 + \boldsymbol{d}\ddot{f}^2 \wedge \boldsymbol{d}f^3 \wedge \boldsymbol{d}f^1 + \boldsymbol{d}\ddot{f}^3 \wedge \boldsymbol{d}f^1 \wedge \boldsymbol{d}f^2 \\ &= \; 3\boldsymbol{d}\ddot{f}^{[1} \wedge \boldsymbol{d}f^2 \wedge \boldsymbol{d}f^{3]} \;\; . \end{aligned}$$

5.3. Vorticity conservation law

Recall that the *vorticity of a flow* is defined by the antisymmetric part of the Eulerian velocity gradient :

$$\omega_{ij} := v_{[i,j]} \;\; . \tag{5.7}$$

Define the corresponding *vorticity two–form* (expressed in the Eulerian basis),

$$\tilde{\boldsymbol{\omega}} := -\omega_{ij}\boldsymbol{d}x^i \wedge \boldsymbol{d}x^j \;\; , \tag{5.8}$$

and prove that the vorticity is conserved in time, i.e.

$$\frac{d}{dt}\tilde{\boldsymbol{\omega}} = \tilde{\mathbf{0}} \;\; , \tag{5.9}$$

with the Lagrangian time–derivative d/dt, if and only if the acceleration field $\mathbf{b}$ is *irrotational*.

Employ first the calculus of differential forms for this proof and compare the result with a tensorial formulation of the evolution equation for the vorticity tensor coefficients ω_{ij} by using Euler's equation.