Formation Sciences de la Matière
Cours : Physique Master 1 (ENS)

Méthodes Mathématiques pour la Physique

TD 6

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In this exercise the goal is to study, in various forms and by combining different methods that we learned thus far, the equations of electrodynamics and one of their extensions, serving at the same time as an example of field theoretical concepts.

We place ourselves into a four–dimensional Minkowski spacetime. Let a point (or its position vector) be characterized by its four (contravariant) coordinates $x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z)$. We denote the partial derivatives according to $\partial_\mu \equiv \partial/\partial x^\mu$. We also make use of the covariant coordinates x_μ, associated with the metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, and the identical inverse matrix $\eta^{\mu\nu}$. Recall that raising and lowering an index just corresponds to changing the sign or not, according to : $u_\mu = \eta_{\mu\mu'} u^{\mu'}$, $u^\mu = \eta^{\mu\mu'} u'_\mu$, $T_{\mu\nu} = \eta_{\mu\mu'}\eta_{\nu\nu'} T^{\mu'\nu'} \dots$.

6.1. Homogeneous equations

We consider the vector potential A^μ, a vector field that is defined at each point x^μ of spacetime through its four real components. We define the 1–form

$$\boldsymbol{\omega} = A_\mu \boldsymbol{d}x^\mu \; .$$

6.1.1. Calculate the exterior derivative $\mathbf{F}$ of $\boldsymbol{\omega}$ and cast it into the form :

$$\mathbf{F} = \boldsymbol{d\omega} = \frac{1}{2} F_{\mu\nu} \boldsymbol{d}x^\mu \wedge \boldsymbol{d}x^\nu \; .$$

Then, express the coefficients $F_{\mu\nu}$ in terms of the partial derivatives of A_μ.

6.1.2. The electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields are classically expressed (in Gaussian units) as $\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial \mathbf{A}/c\partial t$ and $\mathbf{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ as functions of the scalar ϕ and vector $\boldsymbol{A} \equiv (A_x, A_y, A_z)$ potentials. Establish a correspondence between the components of $\boldsymbol{A}$, ϕ, and A_μ (or the A^μ), in order to express $F_{\mu\nu}$ in terms of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

6.1.3. The exterior derivative of $\mathbf{F}$ vanishes since we have $\boldsymbol{d}\mathbf{F} = \boldsymbol{dd\omega} = \mathbf{0}$. Express this fact by employing the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, i.e. find four of the Maxwell equations.

6.2. Equations with sources

We consider the problem in the presence of sources (charge ρ et current $\mathbf{j}$), written with the help of the four–vector of the current density $J^\mu = 4\pi(\rho, \mathbf{j}/c)$. We are interested in the Lagrangian density,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \ ,$$

where the components of the Faraday tensor are given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

6.2.1. Show *with care*, by applying to the action integral a variational principle to the field A^μ, how we can obtain the Euler–Lagrange equations in the following form :

$$\partial_\alpha \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} \right\} = \frac{\partial \mathcal{L}}{\partial A_\beta} \ .$$

6.2.2. Calculate these expressions in order to obtain the relations between the derivatives of the Faraday tensor and the four–vector of the current density.

What is the 4–divergence of J^μ ? Interprete this result.

6.2.3. Translate the preceding equations into the language of Maxwell's relations in terms of the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields.

6.2.4. Coming back to the expressions in terms of the vector potential A^μ, show that employing the Lorenz gauge (in which $\partial_\mu A^\mu = 0$) leads to the equations :

$$\Box A^\mu = J^\mu \ ,$$

where you make the d'Alembert operator $\Box$ explicit.

6.3. The Lagrangian of Proca

We are now interested in the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu + \frac{m^2 c^2}{2\hbar^2} A_\mu A^\mu \ ,$$

where m is strictly positive, and $\hbar$ is Planck's constant.

6.3.1. Apply again the Euler–Lagrange equations. With the condition $\partial_\mu J^\mu = 0$, *show* that $\partial_\mu A^\mu = 0$, and that we obtain the equation of Proca :

$$\left(\Box + \frac{m^2 c^2}{\hbar^2} \right) A^\mu = J^\mu \ .$$

6.3.2. We consider the system under the absence of sources, and look at how the field can propagate. We seek a monochromatic plane–wave solution of the form $A^\mu = \varepsilon^\mu \exp(-i k_\nu x^\nu)$ with the 4–wave vector $k^\mu = (\omega/c, \mathbf{k})$ and the polarization 4–vector ε^μ.

Which constraint exists between ε^μ and k^μ ?

Which dispersion relation $\omega(\boldsymbol{k})$ do we find ? How can we interpret it in relativistic terms, and which meaning can we give to m ?