Formation Sciences de la Matière
Cours : Physique Master 1 (ENS)

Méthodes Mathématiques pour la Physique

TD 7

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7.1. Geometric properties of the acceleration

Show that the acceleration is, in general, not orthogonal to the velocity. Show further that the acceleration always lies within the *osculating plane*, spanned by the unit tangent and unit normal vector fields of a differential curve.

7.2. Curvature of spacetime trajectories

A collection of masses subjected to a one–dimensional gravitational field strength trace curves in the two–dimensional spacetime. Employ the following equations and try to find their general solution giving a family of trajectories. Then, calculate their torsion and curvature.

The evolution of a (conserved) continuous density of masses ϱ that move under a force field density, $\mathbf{f} = \varrho\,\mathbf{g}$, is governed by Euler's equation, the continuity equation, and is subjected to a Newtonian gravitational field equation :

$$\frac{d}{dt}v_1 = g_1 \quad ; \tag{7.1}$$

$$\frac{d}{dt}\varrho = -\varrho\frac{\partial v_1}{\partial x_1} \quad ; \tag{7.2}$$

$$\frac{\partial g_1}{\partial x_1} = -4\pi\varrho \quad . \tag{7.3}$$

7.3. Averaged principal scalar invariants

Show that, by assuming existence of a velocity potential, $\mathbf{v} = \boldsymbol{\nabla} S$, and by performing the spatial average over the principal scalar invariants of the velocity gradient field $(v_{i,j})$, we obtain :

$$\langle II \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} II(v_{i,j}) \; d^3x = \frac{1}{V_{\mathcal{D}}} \int_{\partial\mathcal{D}} H \, |\boldsymbol{\nabla} S|^2 d\sigma \; ; \tag{7.4}$$

$$\langle III \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} III(v_{i,j}) \; d^3x = \pm \frac{1}{V_{\mathcal{D}}} \int_{\partial\mathcal{D}} G \, |\boldsymbol{\nabla} S|^3 d\sigma \quad , \tag{7.5}$$

where H is the local *mean curvature* and G the local *Gaussian curvature* at every point on the $2-$surface bounding the domain. Recall that $|\boldsymbol{\nabla} S| = \frac{ds}{dt}$ with the instrinsic arc length s of the trajectories of fluid elements, and the extrinsic Newtonian time t.

Hint : Make explicit use of the properties of gradient fields and geometrical properties of surfaces appended below.

$$I(v_{i,j}) = \Theta = \boldsymbol{\nabla} \cdot \mathbf{v} \; ; \;\; 2II(v_{i,j}) = \boldsymbol{\nabla} \cdot \Big(\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \Big) \; ; \;\; 3III(v_{i,j}) =$$
$$\boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{\nabla} \cdot \Big(\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \Big) \mathbf{v} - \Big(\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \Big) \cdot \boldsymbol{\nabla} \mathbf{v} \right) . \tag{7.6}$$

At points $P = (x_0, y_0, z_0)$, where the representation of the velocity front in terms of surfaces in the form $z = \chi(x, y)$ is nonsingular, $\boldsymbol{\nabla} S \neq \mathbf{0}$, we have for the mean curvature H and the Gaussian curvature G (Recall that indexed letters denote partial derivatives with respect to the coordinates) :

$$2H := \frac{(1+\chi_x^2)\chi_{yy} - 2\chi_x\chi_y\chi_{xy} + (1+\chi_y^2)\chi_{xx}}{(1+\chi_x^2+\chi_y^2)^{3/2}} \quad ; \tag{7.7}$$

$$G := \frac{\chi_{xx}\chi_{yy} - \chi_{xy}^2}{(1+\chi_x^2+\chi_y^2)^2} \quad . \tag{7.8}$$

Using the implicit definition $S(x, y, \chi(x, y)) = s$ of the velocity front, calculate the derivatives of χ to obtain for the curvature invariants of the front :

$$2H = \tfrac{1}{|\boldsymbol{\nabla} S|^3} \Big[2S_xS_yS_{xy} + 2S_xS_zS_{xz} + 2S_yS_zS_{yz} - S_{xx}(S_y^2 + S_z^2)$$
$$-S_{yy}(S_x^2 + S_z^2) - S_{zz}(S_x^2 + S_y^2) \Big] \quad ; \tag{7.9}$$

$$G = \tfrac{1}{|\boldsymbol{\nabla} S|^4} \Big[S_x^2(S_{yy}S_{zz} - S_{yz}^2) + S_y^2(S_{xx}S_{zz} - S_{xz}^2) + S_z^2(S_{xx}S_{yy} - S_{xy}^2)$$
$$-2S_xS_y(S_{xy}S_{zz} - S_{xz}S_{yz}) - 2S_xS_z(S_{xz}S_{yy} - S_{xy}S_{yz})$$
$$-2S_yS_z(S_{yz}S_{xx} - S_{xy}S_{xz}) \Big] \; . \tag{7.10}$$