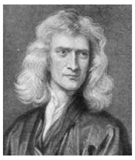
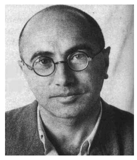



Formation Sciences de la Matière
Cours : Physique Master 2 (ENS)




Cosmologie et Systèmes Gravitationnels

TD 1

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1.1. General integral of the continuity equation

For the density field along trajectories of fluid elements, $\varrho(\mathbf{X},t)$, the following general integral is a solution of the continuity equation for the density as a function of Eulerian coordinates, $\varrho(\mathbf{x},t)$:

$$\varrho(\mathbf{X},t) = \frac{\varrho_o(\mathbf{X})}{J(\mathbf{X},t)} \ . \tag{1.1}$$

Here, J represents the determinant of the Jacobian matrix, $J_{ik}(\mathbf{X},t) \equiv \partial f_i / \partial X_k$, that describes the transformation from Eulerian to Lagrangian coordinates. The initial conditions are given in the form $J(\mathbf{X},t_0) = \det(\delta_{ik}) = 1$ and $\varrho(\mathbf{X},t_0) \equiv \varrho_0(\mathbf{X})$.

Show that, *locally*, the integral (1.1) satisfies the continuity equation :

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \mathbf{v}) = \frac{d\varrho}{dt} + \varrho \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \ . \tag{1.2}$$

With $\mathbf{v}$ we have denoted the velocity field of the fluid elements, and with d/dt the total (or Lagrangian) derivative operator with respect to the time :

$$\frac{d}{dt} \equiv \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \mathbf{v} \cdot \boldsymbol{\nabla} \ . \tag{1.3}$$

Hint : Write the Jacobian J in terms of functional determinants,

$$J = \frac{\partial\,(f_1, f_2, f_3)}{\partial\,(X_1, X_2, X_3)} := \begin{vmatrix} f_{1|1} & f_{1|2} & f_{1|3} \\ f_{2|1} & f_{2|2} & f_{2|3} \\ f_{3|1} & f_{3|2} & f_{3|3} \end{vmatrix} \ , \tag{1.4}$$

where $f_{i|j} := \partial f_i / \partial X_j$, $i = 1 \ldots 3, j = 1 \ldots 3$, and show that

$$\frac{d}{dt} J = J \, \boldsymbol{\nabla} \cdot \mathbf{v} \ . \tag{1.5}$$

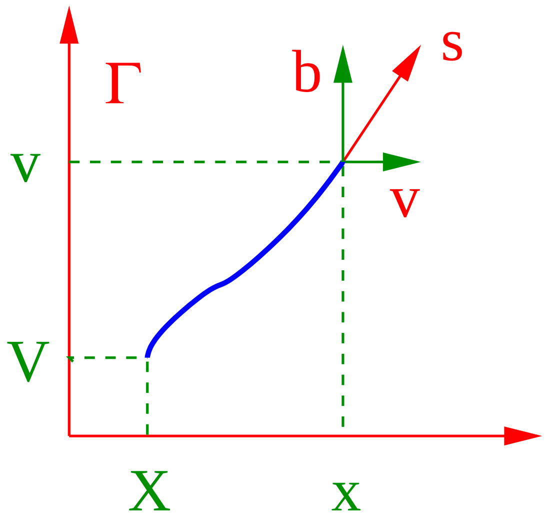

FIGURE 1.1 – The decomposition of the flow vector s into its components *velocity* and *acceleration* is shown in a reduced, two–dimensional section of the one–particle phase space.

1.2. Velocity and acceleration in phase space

Convince yourself that the components of s at a given fluid element are the values of the velocity and the acceleration of this fluid element (Fig. 1.1). Furthermore, convince yourself that, for any function $\eta(\mathbf{x}, t)$, the Lagrangian derivative in phase space reduces to the Lagrangian derivative in space :

$$\frac{D}{Dt}\eta(\mathbf{x}, t) = \frac{d}{dt}\eta(\mathbf{x}, t) \quad . \tag{1.6}$$

Hint : For the first problem write out the Lagrangian derivatives of the orbits from the definition of the phase space flow vector s.

1.3. Incompressible phase space flow

Show that the phase space flow is *incompressible*, i.e., $\nabla_{\mathbf{w}} \cdot \mathbf{s} = 0$, if forces are velocity–independent, $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$. Then, the phase space density $f(\mathbf{w}, t)$ is conserved along the orbits :

$$\frac{D}{Dt}f \;=\; 0 \quad . \tag{1.7}$$

(For a system of N particles this is the contents of Liouville's theorem.) Show further that the converse is not true by giving a counter–example of a velocity–dependent force, for which the phase space flow is also incompressible.

1.4. Remark : Hamilton and Schrödinger representations

In the Hamiltonian picture of *classical mechanics* the independent variables are the (not necessarily Cartesian) canonically conjugate variables $\mathbf{q}$ and $\mathbf{p}$. Choosing $q_i(\mathbf{W}, t) := x_i$ and $p_i(\mathbf{W}, t) := mv_i$, we appreciate that the continuum elements in phase space obey the equations of point mechanics :

$$\frac{D}{Dt}q_i = \frac{1}{m}p_i \quad ; \quad \frac{D}{Dt}p_i = F_i = mg_i \quad . \tag{1.8}$$

Introducing Hamilton's function as a function of the orbit position in phase space (and so implicitly indexed by the Lagrangian coordinates of fluid elements) and explicitly of time,

$$\mathcal{H}(\mathbf{q},\mathbf{p},t) := \frac{1}{2m}\mathbf{p}^2 + m\Phi(\mathbf{q},t) \ \ , \tag{1.9}$$

the evolution equations for the continuum elements can be represented by *Hamilton's equations* :

$$\frac{D}{Dt}q_i = \frac{\partial \mathcal{H}}{\partial p_i} \ \ ; \ \ \frac{D}{Dt}p_i = -\frac{\partial \mathcal{H}}{\partial q_i} \ \ . \tag{1.10}$$

Recall that for an arbitrary phase space function $\xi(\mathbf{q},\mathbf{p},t)$ the total (Lagrangian) time–derivative may be expressed in terms of the Hamiltonian :

$$\frac{D}{Dt}\xi = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial q_i}\frac{Dq_i}{Dt} + \frac{\partial \xi}{\partial p_i}\frac{Dp_i}{Dt} = \frac{\partial \xi}{\partial t} - \{\mathcal{H},\xi\} \ \ , \tag{1.11a}$$

with the *Poisson bracket*

$$\{\mathcal{H},\xi\} := \frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial \xi}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial \xi}{\partial q_i} \ \ . \tag{1.11b}$$

For $\xi = \mathcal{H}$ we have

$$\{\mathcal{H},\mathcal{H}\} \ = \ 0 \ ; \ \ \frac{D\mathcal{H}}{Dt} = \frac{\partial \mathcal{H}}{\partial t} \ \ , \tag{1.12a}$$

so that for $\mathcal{H}(\mathbf{q},\mathbf{p},t) \equiv E(\mathbf{q},\mathbf{p})$ the motion of the fluid is confined to surfaces of constant energy E, $DE/Dt = 0$.

For $\xi = f$ we can use the conservation of the phase space density, $Df/Dt = 0$, to conclude :

$$\frac{\partial f}{\partial t} \ = \ \{\mathcal{H},f\} \ \ . \tag{1.12b}$$

Eq. (1.12b) forms the basis of Jeans' theorems determining *integrals of motion*, if we consider time–independent densities f. These integrals serve, e.g., for the Poincaré analysis of orbits.

There is also a formal analogy to Schrödinger's equation. We write

$$i\,\frac{\partial f}{\partial t} =: \mathcal{B} f \ \ , \tag{1.13a}$$

with the *Boltzmann operator* $\mathcal{B}$ acting on any phase space function :

$$\mathcal{B}\,\xi \ = \ i\,\{\mathcal{H},\xi\} \ \ ; \ \ \xi = f \ \ . \tag{1.13b}$$

This formulation is useful, e.g., for the treatment of scattering problems using the analogy to Schrödinger's equation

$$i\,\frac{\partial}{\partial t}\Psi \ = \ \hat{\mathcal{H}}\Psi \ \ , \tag{1.14}$$

where $\hat{\mathcal{H}}$ denotes the Hamilton operator. We may write the solution of the initial value problem for Vlasov's equation as follows :

$$f(\mathbf{q},\mathbf{p},t) \ = \ e^{-i\mathcal{B}t}\,f_0(\mathbf{q},\mathbf{p}) \ \ , \tag{1.15}$$

where the operator $\mathcal{B}$ is not explicitly time–dependent. This ansatz reproduces the operator equation (1.13a), and $\mathcal{B}$ appears as generator of infinitesimal time–translations. (For further details see, e.g., the books by Saslaw and Balescu.)

1.5. Analogy with Maxwell's equations

Let us have a deeper look at the structure of the field equations by using the analogy to *Maxwell's equations*.

Maxwell's equations for the electric and magnetic field strengths $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ read (we use Gaussian units ; for a clear presentation of classical electrodynamics see Jackson's book) :

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{E} &= -\tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{B} \quad ; \qquad & \boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi\varrho_e \quad ; \\ \boldsymbol{\nabla}\times\mathbf{B} &= \tfrac{4\pi}{c}\mathbf{j}_e + \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{E} \quad ; \qquad & \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \quad , \end{aligned} \tag{1.16}$$

where ϱ_e is the charge density field, and $\mathbf{j}_e := \varrho_e\mathbf{v}$ the charge current density with the associated flow velocity $\mathbf{v}$. The equations for $\mathbf{E}$ are *Faraday's law* and *Coulomb's law*, the equations for $\mathbf{B}$ are *Ampères law* and the law of absence of magnetic monopoles. Recall that with

$$\begin{aligned} \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \Delta\mathbf{E} = -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\nabla}\times\mathbf{B} = -\tfrac{4\pi}{c^2}\tfrac{\partial}{\partial t}\mathbf{j}_e - \tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}\mathbf{E} \;; \\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \Delta\mathbf{B} = \tfrac{4\pi}{c}\boldsymbol{\nabla}\times\mathbf{j}_e - \tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}\mathbf{B} \;, \end{aligned} \tag{1.17}$$

we obtain the following wave equations :

$$\Box\mathbf{E} \;=\; 4\pi\boldsymbol{\nabla}\varrho_e + \frac{4\pi}{c^2}\frac{\partial}{\partial t}\mathbf{j}_e \quad ; \quad \Box\mathbf{B} \;=\; -\frac{4\pi}{c}\boldsymbol{\nabla}\times\mathbf{j}_e \;, \tag{1.18}$$

with the d'Alembertian $\Box := \partial^2/\partial x_0^2 + \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$, written in Minkowski spacetime with coordinates $(x_0 = ict, x_1, x_2, x_3)$.

For vanishing sources, i.e., $\varrho_e = 0$ and therefore $\mathbf{j}_e = \mathbf{0}$, these equations describe the propagation of electromagnetic waves at speed c.

The trivial analogy between the gravitational field equations and Maxwell's equations is obviously to associate $\mathbf{E} \sim -\mathbf{g}$ (setting $\varrho_e = \varrho$, G=1, and $\Lambda = 0$), so that the electrostatic theory would correspond to the gravitational theory. However, this way of comparison is too shortsighted ; we can also find an analogy with the magnetic field strength, as we show now.

Let us consider the field equations for $\mathbf{B}$. Since we also have a current density $\mathbf{j}$ for the flow of fluid elements in the gravitational theory, we may ask about its relation to the time–derivative of the gravitational field strength. Using the continuity equation and the relation between the restmass density and the divergence of the field strength, demonstrate that the following equation holds :

$$\frac{\partial}{\partial t}\mathbf{g} - 4\pi G\mathbf{j} \;=:\; \boldsymbol{\nabla}\times\boldsymbol{\tau} \quad , \tag{1.19}$$

with $\boldsymbol{\tau}$ being the *vector potential* of the current density (while the time–derivative of Φ is the scalar potential) :

$$4\pi G\mathbf{j} \;=\; -\boldsymbol{\nabla}\frac{\partial}{\partial t}\Phi - \boldsymbol{\nabla}\times\boldsymbol{\tau} \quad . \tag{1.20}$$

Since we have no equation for the divergence of $\boldsymbol{\tau}$, we may employ the transverse gauge condition $\boldsymbol{\nabla}\cdot\boldsymbol{\tau} = 0$ to provide the analogy with the magnetic part of Maxwell's equations : $\mathbf{B} \sim -\boldsymbol{\tau}/c$ (setting $\mathbf{j}_\mathrm{e} = \mathbf{j}, G = 1$).

Show that the vector fields $\mathbf{g}$ and $\boldsymbol{\tau}$ obey the Poisson equations :

$$\Delta\mathbf{g} \;=\; -4\pi G\boldsymbol{\nabla}\varrho \quad ; \quad \Delta\boldsymbol{\tau} \;=\; \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\tau}) \;=\; 4\pi G\boldsymbol{\nabla}\times\mathbf{j} \quad . \tag{1.21}$$

In order to obtain some kinematical intuition about the action of $\boldsymbol{\tau}$, we notice that $\boldsymbol{\tau}$ is a harmonic vector field for a class of motions that may be specified by the vanishing of the source in the equation

$$\Delta\boldsymbol{\tau} \;=\; 4\pi G\boldsymbol{\nabla}\times\mathbf{j} = 4\pi G[\;\varrho\boldsymbol{\nabla}\times\mathbf{v}+\boldsymbol{\nabla}\varrho\times\mathbf{v}\;] \quad . \tag{1.22}$$

Try to give the conditions, together with illustrations, of that class of motions for which the vector field $\boldsymbol{\tau}$ is harmonic. A non–trivial $\boldsymbol{\tau}$ causes deviations from such a class of motions.

These equations do not describe gravitational wave propagation. Discuss, by adding the additional source term $-1/c^2\,\partial/\partial t\,\boldsymbol{\tau}$ to the gravitational equations, that this would render the analogy complete and that we obtain gravitational wave equations. Give also the form of the generalized gravitational force, if this term were included.

Notice that including this term into Newton's field equations would render them Lorentz–covariant, i.e. invariant under Lorentz transformations, while without this term they are just Galilei–invariant.

1.6. Remark : interface with plasma physics

Recall the evolution equation for the phase space density $f(\mathbf{w},t)$ in the Eulerian phase space with the independent variables $\mathbf{w}=(\mathbf{x},\mathbf{v})$ and t :

$$\frac{D}{Dt}f+f\boldsymbol{\nabla}_{\mathbf{w}}\cdot\mathbf{s} \;=\; 0 \quad . \tag{1.23}$$

Since this equation is a general consequence of the laws of mechanics for continuum elements in phase space, provided that the acceleration $\mathbf{b}$ is not considered as an independent variable, we can apply it to other physical problems as well. Specifying the forces, $\mathbf{F}=m\mathbf{b}$, renders the kinematical equation (1.23) dynamical.

In Newtonian gravitational systems we have assumed that the forces are not explicitly velocity–dependent, $\mathbf{b}=\mathbf{b}(\mathbf{x},t)$, and therefore the phase space flow is incompressible. However, in our *TD 1.3* we have discussed an example of a velocity–dependent force that also leads to an incompressible phase space flow, obeying the simpler equation :

$$\frac{Df}{Dt}=\frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial x_i}+b_i\frac{\partial f}{\partial v_i}=0 \quad . \tag{1.24}$$

This example is the *Lorentz force* (we use Gaussian units ; c denotes the speed of light),

$$\mathbf{F}^L := q_e(\mathbf{E}+\mathbf{v}\times\mathbf{B}/c) \quad , \tag{1.25}$$

exerted on a moving charge q_e in the presence of electric and magnetic field strengths $\mathbf{E}$ and $\mathbf{B}$. This force is velocity–dependent, but $\partial/\partial v_i\, b_i^L := 1/q_e\,\partial/\partial v_i\, F_i^L = 0$.

Note that in the Lorentz–covariant version of a generalized form of Newton's laws, discussed above, we would also consider a velocity–dependent force.

Considering electromagnetic force fields leads us to the research field of *plasma physics*. We here consider the equation :

$$\frac{Df}{Dt}=\frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial x_i}+\frac{q_e}{m}(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B})_i\frac{\partial f}{\partial v_i}=0 \quad , \tag{1.26}$$

where the fields $\mathbf{E}$ and $\mathbf{B}$ have to obey Maxwell's equations.