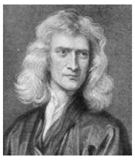
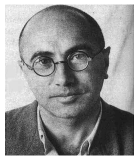



Formation Sciences de la Matière
Cours : Physique Master 2 (ENS)




Cosmologie et Systèmes Gravitationnels

TD 2

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2.1. Distribution function for 'dust'

We learned that the observables

$$\varrho(\mathbf{x},t) = m \int_{\Omega_t^v} f \; d^3v \quad \text{and} \quad \overline{\mathbf{v}}(\mathbf{x},t) = \frac{\int_{\Omega_t^v} \mathbf{v} f \; d^3v}{\int_{\Omega_t^v} f \; d^3v} \tag{2.1}$$

obey the following equations of a continuum in Eulerian space :

$$\frac{d}{dt}\varrho \; + \; \varrho \boldsymbol{\nabla} \cdot \overline{\mathbf{v}} \; = \; 0 \quad ; \quad \frac{d}{dt}\overline{\mathbf{v}} \; = \; \mathbf{b} \quad ; \quad \mathbf{b} = \mathbf{g} \; . \tag{2.2}$$

Show with the help of the definitions (2.1) that the one–particle distribution (peaked around the single–valued velocity field $\overline{\mathbf{v}}$),

$$f^{\rm dust}(\mathbf{x},\mathbf{v},t) = n(\mathbf{x},t) \; \delta(\mathbf{v} - \overline{\mathbf{v}}(\mathbf{x},t)) \;\; , \tag{2.3}$$

describes the matter model *dust*.

2.2. Friedmann equations

Show that for a homogeneous–isotropic trajectory field $\mathbf{f}_H = a(t)\mathbf{X}$ the evolution equation for the *scale–factor* $a(t)$ obeys *Friedmann's differential equations* :

$$3\frac{\ddot{a}}{a} + 4\pi G\varrho_H - \Lambda \;=\; 0 \;\;, \tag{2.4}$$

that can be integrated (by respecting mass–conservation) to give

$$H^2 - \frac{8\pi G}{3}\varrho_H - \frac{\Lambda}{3} + \frac{k}{a^2} \;=\; 0 \;\;, \tag{2.5}$$

with the *Hubble function* $H := \dot{a}/a$, and the constant of integration k.

Derive the solution for an Einstein–de Sitter universe model that is characterized by $\Lambda = 0$ and $k = 0$.

2.3. The Einstein static universe model

Discuss Friedmann's differential equations (2.4, 2.5) for a static universe model by assuming $a(t) =: a_E$, $a_E = const.$ Calculate the "curvature parameter" k and give a_E in terms of k and the constant density $\varrho \equiv \varrho_E$. Discuss the signs of k and Λ. Is there a static model with $\Lambda = 0$?

2.4. Relativistic form of Friedmann's differential equation

In the Newtonian theory we found Friedmann's differential equation (2.5) by integration of the acceleration law (2.4) that followed from Raychaudhuri's equation. In Einstein's theory Raychaudhuri's equation has the same form. However, there is an equation that directly gives the expansion law (2.5) without integration. This equation has no Newtonian analog and it is a constraint that links the kinematical variables and the density to the *intrinsic scalar curvature* of space, $\mathcal{R}$ (which is determined by the Riemannian metric and its first and second space derivatives) :

$$c^2\,\mathcal{R} + \Theta^2 - \Theta^i_{\;j}\,\Theta^j_{\;i} \;=\; 16\pi G\varrho + 2\Lambda \;\;, \tag{2.6}$$

where $\Theta_{ij} := \dot{g}_{ij}$ is the *expansion tensor*, and $\Theta := \Theta^k_{\;k}$ its trace, the *rate of expansion* (which corresponds to the divergence of the velocity field in Newton's theory).

Show that the so–called *Hamilton constraint*, Eq. (2.6), furnishes the relativistic form of Friedmann's differential equation (2.5) in the case of a homogeneous–isotropic matter distribution.

Hints : Employ the expression for a *constant curvature space* $c^2\mathcal{R}_H =: 6k/a(t)^2$, and use the kinematical decomposition of the second scalar invariant of the expansion tensor (here written for vanishing vorticity), $\mathrm{II} := 1/2(\Theta^2 - \Theta^i_{\;j}\,\Theta^j_{\;i}) = 1/3\Theta^2 - \sigma^2$, with the *rate of shear* defined in terms of the trace–free part of the expansion tensor, $\sigma^2 := 1/2\sigma^i_{\;j}\sigma^j_{\;i}$.

2.5. Cosmological parameters

Friedmann's differential equation (2.5) lies at the basis of the *standard model* of cosmology. Commonly, the parameters of the standard model are discussed in terms of the dimensionless *cosmological parameters*. Divide this equation by the square of the Hubble function and define suitable dimensionless parameters that obey

$$\Omega_m + \Omega_\Lambda + \Omega_k \; = \; 1 \; . \tag{2.7}$$

Discuss their representation in terms of the "cosmic triangle" of Figure 2.1.

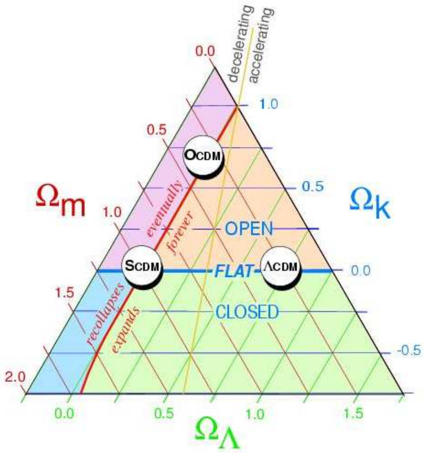

FIGURE 2.1 – Cosmic Triangle.

Can we define them for the Einstein static universe model ?

2.6. Klein–Gordon equation

Consider a scalar field $\Phi(\mathbf{x}^\mu)$ evolving in a FLRW space–time with four–dimensional metric coefficients

$$g_{\mu\nu} = \mathrm{diag}(-1, a^2(t), a^2(t), a^2(t)) \quad ; \quad g^{\mu\nu} = \mathrm{diag}(-1, a^{-2}(t), a^{-2}(t), a^{-2}(t)) \; . \tag{2.8}$$

(we have introduced units where the speed of light $c \equiv 1$).

Introduce the Lorentz–covariant Lagrangian $\mathcal{L}(\partial_\mu \Phi, \Phi, t) := -\frac{1}{2} g^{\mu\nu}(t) \partial_\mu \Phi \partial_\nu \Phi - V(\Phi)$, with kinetic and potential terms and an explicit function of the time–parameter t. Show that the corresponding least action principle yields the relativistic wave equation (an overdot denotes partial time–derivative) :

$$\ddot{\Phi} + 3H\dot{\Phi} - \frac{1}{a^2}\Delta\Phi + \frac{\partial V(\Phi)}{\partial \Phi} = 0 \;\; , \tag{2.9}$$

with $H(t) := \dot{a}/a$. Discuss the role of the function $a(t)$.

Now, specify the problem to a spatially constant scalar field $\Phi(t)$, and write down the evolution equation for the specific form of the *Ginzburg–Landau potential* :

$$V^{\mathrm{GL}} = V_0 \frac{(\Phi^2 - \Phi_0^2)^2}{\Phi_0^4} \;\; , \tag{2.10}$$

where Φ_0 is some initial value for the scalar field.