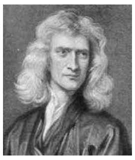
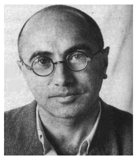



# Formation Sciences de la Matière
# Cours : Physique Master 2 (ENS)

*Cosmologie et Systèmes Gravitationnels*

## TD 4

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### 4.1. Evolution equation for $\Pi_{ij}$

Derive from the following balance equation – given in the course –,

$$\frac{\partial}{\partial t}\left(\varrho\overline{v_i v_j}\right) + \frac{\partial}{\partial x_k}\left(\varrho\overline{v_i v_j v_k}\right) \;=\; \varrho\left(g_i\overline{v}_j + g_j\overline{v}_i\right) \quad , \tag{4.1}$$

the evolution equation for the reduced second moment tensor $\Pi_{ij}$,

$$\Pi_{ij} := \varrho\left(\overline{v_i v_j} - \overline{v}_i\overline{v}_j\right) \;=\; \varrho\overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} \quad , \tag{4.2}$$

(a comma abbreviates spatial partial derivative with respect to Eulerian coordinates) :

$$\frac{d}{dt}\Pi_{ij} = -\left[\,\overline{v}_{k,k}\Pi_{ij} + \overline{v}_{i,k}\Pi_{kj} + \overline{v}_{j,k}\Pi_{ki} + L_{ijk,k}\,\right] \quad , \tag{4.3a}$$

with the *reduced third velocity moment*

$$\begin{aligned} L_{ijk} &:= \varrho\overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)(v_k - \overline{v}_k)} \\ &= \varrho\left(\overline{v_i v_j v_k} - \overline{v}_i\overline{v}_j\overline{v}_k\right) - \left[\,\overline{v}_i\Pi_{jk} + \overline{v}_j\Pi_{ki} + \overline{v}_k\Pi_{ij}\,\right] \quad . \end{aligned} \tag{4.3b}$$

Show in particular that the second equality in the above equation holds. Show further that $L_{ijk}$ is a symmetric tensor.

### 4.2. General formula for the velocity moment hierarchy

Start by deriving the evolution equation for the reduced third velocity moment $L_{ijk}$ by taking the third moment of Vlasov's equation that links the third and fourth velocity moments of $f$. Then employ the Lagrangian derivative and the previous evolution equations of the hierarchy to derive the following evolution equation :

$$\frac{d}{dt}L_{ijk} = -\left[\, \overline{v}_{\ell,\ell}L_{ijk} + \overline{v}_{k,\ell}L_{\ell ij} + \overline{v}_{i,\ell}L_{\ell jk} + \overline{v}_{j,\ell}L_{\ell ki} + M_{ijk\ell,\ell} \,\right] + \frac{1}{\varrho}\left(\Pi_{ij}\Pi_{k\ell,\ell} + \Pi_{jk}\Pi_{i\ell,\ell} + \Pi_{ki}\Pi_{j\ell,\ell}\right) \quad , \tag{4.4a}$$

with the symmetric *reduced fourth velocity moment*

$$M_{ijk\ell} := \varrho\overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)(v_k - \overline{v}_k)(v_\ell - \overline{v}_\ell)} \quad . \tag{4.4b}$$

(You may even try to find the general formula for the whole hierarchy, but this task is very challenging !)

### 4.3. Klimontovich's equation and Liouville's theorem

Show that, as a result of Hamilton's equations for point masses, the *Klimontovich density* $f^K$,

$$f^K(\mathbf{x},\mathbf{v},t) := \sum_{\ell=1}^{N} \delta(\mathbf{x} - \mathbf{x}^{(\ell)})\delta(\mathbf{v} - \mathbf{v}^{(\ell)}) \quad , \tag{4.5}$$

obeys the *Klimontovich equation* :

$$\frac{Df^K}{Dt} = 0 \quad \Leftrightarrow \quad \frac{\partial f^K}{\partial t} + v_i\frac{\partial f^K}{\partial x_i} + g_i\frac{\partial f^K}{\partial v_i} = 0 \quad . \tag{4.6}$$

Discuss the physical meaning of this equation for an N–particle system.

### 4.4. Dynamical equation of state

Integrate the equation

$$\frac{1}{p}\frac{dp}{dt} = \frac{5}{3}\frac{1}{\varrho}\frac{d\varrho}{dt} \quad ; \quad \sigma_{ij} = 0 \quad , \tag{4.7}$$

for $p$ and $\varrho$ using the *Lagrange method* to obtain the dynamical equation of state along trajectories of fluid elements. Discuss how we could interpret this solution as a global equation of state of the fluid.