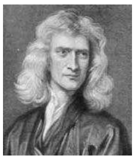
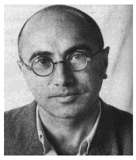


Formation Sciences de la Matière
Cours : Physique Master 2 (ENS)



Cosmologie et Systèmes Gravitationnels

TD 5

Thomas Buchert, CRAL, ENS de Lyon — thomas.buchert@ens–lyon.fr
Pierre Mourier, CRAL, ENS de Lyon — pierre.mourier@ens–lyon.fr

---

5.1. Vorticity transport equations and theorems

If our system develops under the action of a potential force, which is the case for a self–gravitating Newtonian *dust* continuum, then we have from Euler's equation $d/dt\,\overline{\mathbf{v}} = \mathbf{g}$:

$$\boldsymbol{\nabla} \times \mathbf{g} = \boldsymbol{\nabla} \times \left[\frac{\partial}{\partial t}\overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \boldsymbol{\nabla})\overline{\mathbf{v}} \right] = \mathbf{0} \ . \tag{5.1}$$

First, employ the vector identities

$$2(\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b} = \boldsymbol{\nabla} \times (\mathbf{b} \times \mathbf{a}) + \boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{b}) + \mathbf{a}\boldsymbol{\nabla} \cdot \mathbf{b} - \mathbf{b}\boldsymbol{\nabla} \cdot \mathbf{a} - \mathbf{a} \times (\boldsymbol{\nabla} \times \mathbf{b}) - \mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{a}) \ , \tag{5.2a}$$

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b} + \mathbf{a}\boldsymbol{\nabla} \cdot \mathbf{b} - \mathbf{b}\boldsymbol{\nabla} \cdot \mathbf{a} \ , \tag{5.2b}$$

for any vectors $\mathbf{a}$, $\mathbf{b}$, in order to cast Eq. (5.1) into an evolution equation for the *vorticity* $\boldsymbol{\omega} := 1/2\,\boldsymbol{\nabla} \times \overline{\mathbf{v}}$, using the first identity :

$$\frac{\partial}{\partial t}\boldsymbol{\omega} + \boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \overline{\mathbf{v}}) = \mathbf{0} \ , \tag{5.3a}$$

and, using the second identity :

$$\frac{d}{dt}\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\overline{\mathbf{v}} - \boldsymbol{\omega}\boldsymbol{\nabla} \cdot \overline{\mathbf{v}} \ . \tag{5.3b}$$

We have written the latter equation with the help of the Lagrangian derivative. Eq. (5.3b) is known as the *Helmholtz transport equation* for the vorticity. Upon eliminating the divergence of the mean velocity field through the continuity equation, confirm that we can also write it as follows :

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\varrho}\right) = \frac{\boldsymbol{\omega}}{\varrho} \cdot \boldsymbol{\nabla}\overline{\mathbf{v}} \ , \tag{5.4}$$

which is known as *Beltrami's transport equation*. It is interesting that we can find an exact integral of this equation, as derived in the following *TD*.

5.2. Cauchy's integral

Show that, along integral curves $\mathbf{x} = \mathbf{f}(\mathbf{X}, t)$ of the mean velocity field,

$$\left(\frac{\boldsymbol{\omega}}{\varrho}\right)(\mathbf{X}, t) \;=\; \left(\frac{\boldsymbol{\omega}}{\varrho}\right)(\mathbf{X}, t_0) \cdot \boldsymbol{\nabla}_0 \mathbf{f}(\mathbf{X}, t) \tag{5.5}$$

is a general integral of Eq. (5.4); $\boldsymbol{\nabla}_0$ denotes the nabla operator with respect to the Lagrangian coordinates $\mathbf{X}$.

5.3. Vorticity evolution equation for pressure–supported fluids

Start with the *Euler–Jeans equation* for isotropic stresses,

$$\boldsymbol{\nabla} \times \mathbf{g} = \boldsymbol{\nabla} \times \left[\frac{\partial}{\partial t}\overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \boldsymbol{\nabla})\overline{\mathbf{v}} + \frac{1}{\varrho}\boldsymbol{\nabla} p \right] = \mathbf{0} \;\; , \tag{5.6}$$

and employ the vector identities (5.2a) and (5.2b), and additionally

$$\boldsymbol{\nabla} \times \alpha\mathbf{a} = \alpha\boldsymbol{\nabla} \times \mathbf{a} + \boldsymbol{\nabla}\alpha \times \mathbf{a} \;\; , \tag{5.7}$$

for any scalar α, in order to cast Eq. (5.6) into an evolution equation for the *vorticity* $\boldsymbol{\omega} := 1/2\boldsymbol{\nabla} \times \overline{\mathbf{v}}$, as we have already done in the case of a *dust* continuum above. Find the equations of Helmholtz and Beltrami with pressure, and try to integrate the latter along trajectories of fluid elements.

5.4. Transformation of a model equation

Show that the following time–dependent transformation of $\mathbf{v}$ and the change of the time–variable, $t \to \tau$,

$$\mathbf{v} =: \alpha(t)\boldsymbol{\nu} \;\; ; \;\; \frac{\dot{\alpha}}{\alpha} = h(t) \;\; ; \;\; \dot{\tau} = \alpha \;\; , \tag{5.8}$$

leads to the model equation :

$$\frac{d}{d\tau}\boldsymbol{\nu} = \frac{\partial}{\partial \tau}\boldsymbol{\nu} + \boldsymbol{\nu} \cdot \boldsymbol{\nabla}\boldsymbol{\nu} = \mu(\varrho, t)\Delta\boldsymbol{\nu} \quad \text{with} \quad \mu := \frac{1}{\varrho}\frac{\partial \beta}{\partial \varrho}\frac{h(t)}{4\pi G\alpha(t)} \;\; . \tag{5.9}$$

For $\mu = \mathrm{const.}$ this equation is known as the *three–dimensional Burgers equation*. In general, the coefficient in front of the Laplacian is density– and time–dependent. Notwithstanding, *Burgers' equation* is still a good model in many circumstances.