A SIMPLE INFINITE QUANTUM GRAPH

SEBASTIAN ENDRES AND FRANK STEINER

ABSTRACT. We study the Schrödinger equation on an infinite metric graph where the Hamiltonian is given by a suitable one-dimensional Dirichlet Laplacian. The metric structure is defined by assigning an interval $I_n = [0, l_n]$, $n \in \mathbb{N}$, to each edge of the graph with $l_n = \frac{\pi}{n}$. The spectrum of this system is purely discrete with the eigenvalues given by $\lambda_n = n^2$, $n \in \mathbb{N}$, with multiplicities $d(n)$, where $d(n)$ denotes the divisor function. We thus can relate the spectral problem of this infinite quantum graph to Dirichlet's famous divisor problem and infer the non-standard Weyl asymptotics $\mathcal{N}(\lambda) = \frac{\sqrt{\lambda}}{2} \ln \lambda + \mathrm{O}\left(\sqrt{\lambda}\right)$ for the eigenvalue counting function. Based on an exact trace formula, we derive explicit formulae for the trace of the wave group, the heat kernel, the resolvent and for various spectral zeta functions. These results enable us to establish a well-defined (renormalized) secular equation.

1. The quantum mechanical setting

Our infinite quantum graph Γ consists of infinitely many edges e_n, $n = 1 \ldots \infty$, assigned with the intervals $\mathrm{I}_n = \left[0, \frac{\pi}{n}\right]$. Therefore, the total length $\mathcal{L}$ of the quantum graph is infinite and thus it is a noncompact quantum graph. We don't specify the topology of the quantum graph. The corresponding Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ consists of the vector space

$$\mathcal{H} := \left\{ \psi \in \bigoplus_{n=1}^{\infty} \mathrm{L}^2\left[0, \tfrac{\pi}{n}\right]; \quad \|\psi\| := \langle\psi,\psi\rangle_{\mathcal{H}}^{\frac{1}{2}} < \infty \right\} \tag{1.1}$$

equipped with the scalar product

$$\langle\psi,\phi\rangle_{\mathcal{H}} := \sum_{n=1}^{\infty} \langle\psi_n,\phi_n\rangle_{\mathrm{L}^2\left[0,\frac{\pi}{n}\right]} \;, \quad \psi = \bigoplus_{n=1}^{\infty} \psi_n \;, \quad \phi = \bigoplus_{n=1}^{\infty} \phi_n \quad \text{and} \quad \psi_n, \phi_n \in \mathrm{L}^2\left[0, \tfrac{\pi}{n}\right] \quad . \tag{1.2}$$

We define the Laplace operator with Dirichlet boundary conditions $(\Delta, \mathrm{D}(\Delta))$ on Γ as

$$\Delta\psi := \bigoplus_{n=1}^{\infty} \Delta_n \psi_n \;, \quad \psi \in \mathrm{D}(\Delta) \tag{1.3}$$

and the domain of definition $\mathrm{D}(\Delta)$ of Δ as

$$\mathrm{D}(\Delta) := \left\{ \psi \in \mathcal{H}; \quad \psi_n \in \mathrm{H}^2\left[0, \tfrac{\pi}{n}\right] \quad \text{and} \quad \psi_n(0) = \psi_n\left(\tfrac{\pi}{n}\right) = 0 \right\} \quad , \tag{1.4}$$

where in (1.3) we denote with Δ_n the common second weak derivative operator acting on $\mathrm{H}^2\left[0, \frac{\pi}{n}\right]$ being the Sobolev space consisting of all functions in $\mathrm{L}^2\left[0, \frac{\pi}{n}\right]$ for which the first and the second weak derivatives are also elements of $\mathrm{L}^2\left[0, \frac{\pi}{n}\right]$ (see e.g. [1, 2]). In (1.4) we denote with ψ_n the corresponding vector in the orthogonal decomposition of ψ as in (1.2).

Since the quantum graph Γ consists of infinitely many edges, we cannot directly apply the results of [3, 4, 5, 6, 1]. However, by a rearrangement of the rows and columns of the matrices some results still hold but unfortunately not all. We shall discuss this in section 6 in more detail.

We observe that due to the Dirichlet boundary conditions on all interval ends there is no "interaction" between the edges. The S-matrix S of $(-\Delta, \mathrm{D}(\Delta))$ corresponds to the identity operator $\mathbb{1}$ on l^2 being the set of all square summable sequences of real numbers. For compact quantum graphs with Dirichlet boundary conditions the assigned quantum graph as discussed in [5] (the unique quantum graph with maximal vertex numbers for which the S-matrix is local) is simply the set of edges with no common vertex for two different interval ends. This corresponds to a "hard wall reflection" of the particle at the interval ends (see [2]). But for our system the requirement of a maximal number of vertices for which the S-matrix is local does not make sense since there are many different topologies with infinitely countable many vertices (e.g. a star graph). However, the "hard wall" interpretation of the Dirichlet boundary conditions still holds.

For this simple system the stationary Schrödinger equation

$$-\Delta\psi = \lambda\psi \ , \quad \psi \in \mathrm{D}(\Delta) \tag{1.5}$$

is explicitly solvable. The normalized eigenvectors $\psi_{n,m}$ are expressed by the orthogonal decomposition as in (1.2) by

$$\mathrm{P}_l\left(\psi_{n,m}\right)(x) := \delta_{ln}\sqrt{\frac{2n}{\pi}}\sin\left(k_{n,m}x\right) \quad \text{with} \quad k_{n,m} := nm \ , \quad l,n,m \in \mathbb{N} \quad , \tag{1.6}$$

where we have introduced the projector P_l from $\mathcal{H}$ onto $\mathrm{L}^2\left[0,\frac{\pi}{l}\right]$ (see (1.2)) and the wave numbers $k_{n,m} := +\sqrt{\lambda_{n,m}}$. Since the set $\{\psi_{n,m}; \quad m = 1,\ldots,\infty\}$ (n fixed) forms an orthonormal basis of $\mathrm{L}^2\left[0,\frac{\pi}{n}\right]$, we immediately infer that the eigenvectors $\psi_{n,m}$ form a complete orthonormal basis of $\mathcal{H}$. Therefore, the spectrum $\sigma\left((-\Delta,\mathrm{D}(\Delta))\right)$ is purely discrete (see e.g. [7]) and the set of the corresponding wave numbers is identical with the set of the natural numbers $\mathbb{N}$. The multiplicity of the eigenvalue $\lambda = k^2$ respectively of the corresponding wave number k is given by $d(k)$, i.e. it is the divisor function defined as ($k \in \mathbb{N}$)

$$d(k) := \#\left\{(n,m); \quad nm = k\right\} \quad . \tag{1.7}$$

Note that the divisor functoin $d(n)$, $d(1) = 1$, $d(2) = 2$, $d(3) = 2$, $d(4) = 3$, $d(5) = 2$, $d(6) = 4$, $\ldots$, with $d(p) = 2$ for p prime, is a very irregular function with asymptotic behaviour

$$d(n) = \mathrm{O}\left(n^{\epsilon}\right) \ , \quad n \to \infty \ , \quad \epsilon > 0 \quad . \tag{1.8}$$

2. Spectral asymptotics and trace formula

From section 1 and in particular (1.7) we conclude that the number of wave numbers of (1.5) less than x, the spectral counting function $N(x)$, is closely related to the Dirichlet divisor problem. Explicitly, one obtains

$$N(x) := \#\left\{(n,m); \quad \sqrt{\lambda_{n,m}} = nm \leq x\right\} = \sum_{n\leq x} d(n) \ , \quad n \in \mathbb{N} \quad . \tag{2.1}$$

The famous divisor problem of Dirichlet is that of determining the asymptotic behaviour of $N(x)$ as $x \to \infty$. Reducing the problem to a lattice point problem, i.e. counting the positive integer lattice points under the hyperbola $n \leq \frac{x}{m}$, Dirichlet proved [8]

$$N(x) = x\ln x + (2\gamma - 1)\,x + \Delta(x) \ , \quad \Delta(x) = \mathrm{O}\left(x^{\frac{1}{2}}\right) \quad , \tag{2.2}$$

wherein γ denotes the Euler constant. From (2.2) it follows that the average order of the multiplicities g of the corresponding wave numbers $k_{n,m}$ respectivley the eigenvalues $\lambda_{n,m}$ of our infinite quantum graph is given by

$$\overline{g}(x) := \frac{1}{x}\sum_{n\leq x} d(n) = \ln x + (2\gamma - 1) + \mathrm{O}\left(\frac{1}{\sqrt{x}}\right) \ , x \to \infty \quad . \tag{2.3}$$

Furthermore, we deduce that the counting function of the eigenvalues $\lambda_{n,m} = k_{n,m}^2 = n^2m^2$,

$$\mathcal{N}(\lambda) := \#\left\{(n,m); \quad \lambda_{n,m} \leq \lambda\right\} \tag{2.4}$$

is given by $\mathcal{N}(\lambda) = N\left(\sqrt{\lambda}\right)$, i.e. we derive from Dirichlet's result (2.2) the modified Weyl's law

$$\mathcal{N}(\lambda) = \frac{\sqrt{\lambda}}{2}\ln\lambda + (2\gamma - 1)\sqrt{\lambda} + \mathrm{O}\left(\lambda^{\frac{1}{4}}\right) \ , \quad \lambda \to \infty \quad . \tag{2.5}$$

This should be compared with the counting function for a compact quantum graph with total length $\mathfrak{L} < \infty$, for which one has the standard Weyl asymptotics $\mathcal{N}(\lambda) = \frac{\mathfrak{L}}{\pi}\sqrt{\lambda} + \mathrm{O}(1)$ [1, 2].

The estimate on $\Delta(x)$ was later improved i.a. by [9] ($\Delta(x) = \mathrm{O}\left(x^{\frac{1}{3}} \ln x\right)$) and [10] ($\Delta(x) = \mathrm{O}\left(x^{\Theta}\right)$ with $\Theta < \frac{33}{100}$). In [11, 12] it was proven that $\Delta(x) = \mathrm{O}\left(x^{\Theta}\right)$ with $\Theta \geq \frac{1}{4}$, but the exact order of $\Delta(x)$ is still unknown.

It turns out that the Voronï summation formula [13] can be interpreted as a trace formula for the noncompact quantum graph with pure Dirichlet boundary conditions specified in (1.1) and (1.4). In various articles (i.a. [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]) the authors investigate the Voronoï summation formula and specify proper function spaces for which the Voronoï summation formula is valid. For our purpose, we use a result of [19, 20] implementing that, due to the imposed Dirichlet boundary conditions for the quantum graph, the length spectrum of the primitive periodic orbits is precisely given by the set of numbers $\left\{\mathfrak{l}_{p,n} := \frac{2\pi}{n}; \quad n = 1, \ldots, \infty\right\}$ and the wave numbers k_n of the quantum graph are exactly given by the natural numbers $\mathbb{N}$ with corresponding multiplicities $g_n = d(n)$.

Theorem 2.1 (Wilton:1932, Dixon/Ferrar:1937, [19, 20])**.** *If, for any finite t_0, $f(t)$ is a real function of bounded variation in the interval $(0, t_0)$ and is continuous at $t = 1, 2, 3, \ldots$, then*

$$\begin{aligned}\sum_{n=1}^{\infty} g_n f(k_n) &= \int_0^{\infty} (\ln t + 2\gamma)\, f(t) \mathrm{d}t + \frac{f(0)}{4} \\ &+ 2\pi \sum_{n=1}^{\infty} d(n) \int_0^{\infty} \left[\frac{2}{\pi} K_0\left(2(2\pi)^{\frac{3}{2}} \sqrt{\frac{t}{\mathfrak{l}_{p,n}}}\right) - Y_0\left(2(2\pi)^{\frac{3}{2}} \sqrt{\frac{t}{\mathfrak{l}_{p,n}}}\right)\right] f(t) \mathrm{d}t \quad ,\end{aligned} \tag{2.6}$$

provided that

- $\left(V_{0+}^{x} f(t)\right) \ln x \to 0$ *as* $x \to 0^+$,
- *for some positive* κ, $t^{\frac{1}{2}+\kappa} f(t) \to 0$ *as* $t \to \infty$,
- $f(t)$ *is the (indefinite) integral of* $f'(t)$ *in* $t \geq t_0$,
- *for some positive* κ

$$\int_{\kappa}^{\infty} t^{\frac{1}{2}+\kappa} \left|f'(t)\right| \mathrm{d}t < \infty \quad . \tag{2.7}$$

In (2.6) the functions $K_0(x)$ and $Y_0(x)$ are modified Bessel functions, i.e. the McDonald function respectively the Neumann function (see e.g [24]) and $V_{0+}^{x} f(t)$ denotes the total variation of $f(t)$ in $(0, x)$ (see e.g.[25] for a precise definition). In the sequel we investigate the trace of the wave group (however, in "euclidean" time $t \to \mathrm{i}t$), the trace of the heat kernel, the trace of the resolvent, some special zeta functions and a "renormalized" secular equation.

3. The trace of the wave group

We shall need results from asymptotic analysis. We refer to [26] p.320 for the following proposition.

Proposition 3.1. *Let $f : (0, \infty) \to \mathbb{C}$ be a smooth function, which together with all its derivatives is of sufficient decay at infinity (e.g. $f^{(n)}(x) = \mathrm{O}\left(\frac{1}{x^{1+\epsilon}}\right)$ for all $n \in \mathbb{N}_0$) and possesses the asymptotic expansion*

$$f(x) \sim \sum_{n=0}^{\infty} b_{\lambda_n} x^{\lambda_n} \ , \quad x \to 0 \quad , \tag{3.1}$$

with $-1 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$, then the function

$$g(x) := \sum_{n=1}^{\infty} f(nx) \tag{3.2}$$

possesses the asymptotic expansion

$$g(x) := \frac{1}{x}\left(\mathrm{I}_f^* - b_{-1}\ln x\right) + \sum_{n=1}^{\infty} b_{\lambda_n}\zeta(-\lambda_n)(-x)^{\lambda_n} \ , \quad x \to 0^+ \quad , \tag{3.3}$$

with

$$\mathrm{I}_f^* := \int_0^\infty \left(f(t) - b_{-1}\frac{\mathrm{e}^{-t}}{t}\right)\mathrm{d}t \quad . \tag{3.4}$$

Here $\zeta(z)$ in (3.3) denotes the Riemann zeta function.

Using for a certain Lambert series the identity ([27] p.467)

$$\sum_{n=1}^{\infty} d(n)q^n = \sum_{n=1}^{\infty}\frac{q^n}{1-q^n} \ , \quad |q| < 1 \quad , \tag{3.5}$$

we obtain for the trace $\Theta_{\Delta_D^{\frac{1}{2}}}(t)$ of the (euclidean) wave group $\mathrm{e}^{-t\sqrt{-\Delta_D}}$ with $-\Delta_D := (-\Delta, \mathrm{D}(\Delta))$ (see section 1)

$$\Theta_{\Delta_D^{\frac{1}{2}}}(t) := \mathrm{Tr}\left(\mathrm{e}^{-t\sqrt{-\Delta_D}}\right) = \sum_{n=1}^{\infty} d(n)\mathrm{e}^{-nt} = \sum_{n=1}^{\infty}\frac{1}{\mathrm{e}^{nt}-1} = \sum_{n=1}^{\infty} f(nt) \ , \quad t > 0 \quad , \tag{3.6}$$

where we have defined $f(t) := \frac{1}{\mathrm{e}^t - 1}$. It is obvious that $f(t)$ fulfills the requirements of proposition 3.1. Furthermore, for $f(t)$ we have the Laurent series (see e.g. [26] p.278)

$$f(t) = \sum_{k=-1}^{\infty}\frac{B_{k+1}}{(k+1)!}t^k \ , \quad 0 < |t| < 2\pi \quad , \tag{3.7}$$

where B_k are the Bernoulli numbers ($B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$ and $B_3 = B_5 = B_7 = \ldots = 0$). For the calculation of

$$\mathrm{I}_f^* := \int_0^\infty \left[\frac{1}{\mathrm{e}^t - 1} - \frac{\mathrm{e}^{-t}}{t}\right]\mathrm{d}t \tag{3.8}$$

we use two identities, and by a comparison argument we deduce the value of I_f^*. The first identity is ([24] p.20)

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} + \frac{1}{\Gamma(z)}\int_0^\infty \mathrm{e}^{-t}\left[\frac{1}{\mathrm{e}^t-1} - \frac{1}{t} + \frac{1}{2}\right]t^{z-1}\mathrm{d}t \ , \quad \mathrm{Re}\, z > -1 \quad . \tag{3.9}$$

Using the Taylor expansions of t^{z-1} and the reciprocal gamma function $\Gamma(z)^{-1}$ ([24] p.2) at $z = 1$, we obtain by (3.9)

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} + \int_0^\infty \mathrm{e}^{-t}\left[\frac{1}{\mathrm{e}^t-1} - \frac{1}{t} + \frac{1}{2}\right]\mathrm{d}t + \mathrm{O}(z-1) \quad . \tag{3.10}$$

On the other hand it holds the identity ([24] p.19)

$$\zeta(z) = \frac{1}{z-1} + \gamma + \mathrm{O}(z-1) \quad , \tag{3.11}$$

where γ is the Euler constant. Thus, we infer $\mathrm{I}_f^* = \gamma$ and get by proposition 3.1 and the identity $\zeta(-m) = -\frac{B_{m+1}}{m+1}$ for $m = 1, 2, 3, \ldots$ ([24] p.19) the asymptotic expansion

$$\Theta_{\Delta_D^{\frac{1}{2}}}(t) = -\frac{\ln t}{t} + \frac{\gamma}{t} + \sum_{m=0}^{\infty}\frac{B_{m+1}^2}{(m+1)(m+1)!}(-t)^m \ , \quad t \to 0^+ \quad . \tag{3.12}$$

The estimate $\Theta_{\Delta_D^{\frac{1}{2}}}(t) = \mathrm{O}\left(\mathrm{e}^{-t}\right)$ for $t \to \infty$ is trivial.

Note that from the small-t asymptoitcs (3.12) one derives the leading asymptotic term for the counting function (2.2) using the Karamata-Tauberian theorem in the form [28]

$$\lim_{t\to 0^+}\left[-\frac{t^r}{\ln t}\,\mathrm{Tr}\,\mathrm{e}^{-t\sqrt{-\Delta_D}}\right]=c \quad \text{iff} \quad \lim_{x\to\infty}\frac{N(x)}{x^r\ln x}=\frac{c}{\Gamma(r+1)} \quad . \tag{3.13}$$

The function $h_t(p):=\mathrm{e}^{-tp}$, $t>0$, fulfills the requirements in theorem 2.1, and we get for the l.h.s in (2.6) exactly the trace of the wave group $\mathrm{e}^{-t\sqrt{-\Delta_D}}$ (3.6). Therefore, we evaluate the r.h.s in (2.6) and obtain

$$\Theta_{\Delta_D^{\frac12}}(t):=\Theta^W_{\Delta_D^{\frac12}}(t)+\Theta^{Osc.}_{\Delta_D^{\frac12}}(t) \quad , \tag{3.14}$$

where we have defined a "Weyl" term

$$\begin{aligned}\Theta^W_{\Delta_D^{\frac12}}(t) &:= \int\limits_0^\infty(\ln p+2\gamma)h_t(p)\mathrm{d}p+\frac{h_t(0)}{4}\\ &= \int\limits_0^\infty \mathrm{e}^{-tp}\ln p\mathrm{e}^{-tp}\mathrm{d}p+2\gamma\int\limits_0^\infty \mathrm{e}^{-tp}\mathrm{d}p+\frac14\\ &= -\frac{\ln t}{t}+\frac{\gamma}{t}+\frac14\ ,\quad t\to 0^+ \quad ,\end{aligned} \tag{3.15}$$

and an "oscillatory" term

$$\begin{aligned}\Theta^{Osc.}_{\Delta_D^{\frac12}}(t) &:= 2\pi\sum_{n=1}^\infty d(n)\int\limits_0^\infty\left[\frac{2}{\pi}K_0\left(4\pi\sqrt{np}\right)-Y_0\left(4\pi\sqrt{np}\right)\right]h_t(p)\mathrm{d}p\\ &= 4\pi\sum_{n=1}^\infty d(n)\int\limits_0^\infty\left[\frac{2}{\pi}K_0\left(4\pi\sqrt{n}y\right)-Y_0\left(4\pi\sqrt{n}y\right)\right]\mathrm{e}^{-ty^2}y\mathrm{d}y\\ &= -\frac{2}{t}\sum_{n=1}^\infty d(n)\left[\exp\left(\frac{4\pi^2 n}{t}\right)\mathrm{Ei}\left(-\frac{4\pi^2 n}{t}\right)+\exp\left(-\frac{4\pi^2 n}{t}\right)\mathrm{Ei}\left(\frac{4\pi^2 n}{t}\right)\right] \quad ,\end{aligned} \tag{3.16}$$

where the integrals in the second line can be found in [29] p.352 respectively [29] p.266 and $\mathrm{Ei}(x)$ is the exponential integral function. From its asymptotics for $x\to\infty$ ([24] p.346, 347) we conclude

$$e^x\,\mathrm{Ei}(-x)+e^{-x}\,\mathrm{Ei}(x)=\mathrm{O}\left(\frac{1}{x^2}\right)\ ,\quad x\to\infty \quad . \tag{3.17}$$

Therefore, we get for the "oscillatory" term (3.16) the asymptotics ($x=\frac{4\pi^2 n}{t}$)

$$\Theta^{Osc.}_{\Delta_D^{\frac12}}(t)=\mathrm{O}(t)\sum_{n=1}^\infty\frac{d(n)}{n^2}=\mathrm{O}(t)\zeta^2(2)=\mathrm{O}(t)\ ,\quad t\to 0^+ \quad . \tag{3.18}$$

We thus infer from (3.14), (3.15) and (3.18)

$$\Theta_{\Delta_D^{\frac12}}(t)=-\frac{\ln t}{t}+\frac{\gamma}{t}+\frac14+\mathrm{O}(t)\ ,\quad t\to 0^+ \tag{3.19}$$

in complete agreement with (3.12).

Finally, we define the generalized "spectral zeta function" for $\sqrt{-\Delta_D}$ (motivated by [30])

$$\zeta_{\Delta_D^{\frac12}}(s,k):=\mathrm{Tr}\left[\sqrt{-\Delta_D}+k\right]^{-s}=\sum_{n=1}^\infty\frac{d(n)}{(n+k)^s}\ ,\quad \mathrm{Re}\,s>1\ ,\quad k\neq -n\ ,\quad n\in\mathbb{N} \quad , \tag{3.20}$$

(defined by the principal branch of the logarithm) which is the Mellin transfom of

$$\Theta_{\Delta_D^{\frac12}}(t,k):=\sum_{n=1}^\infty d(n)\mathrm{e}^{-(n+k)t}=\mathrm{e}^{-kt}\Theta_{\Delta_D^{\frac12}}(t)\ ,\quad t>0\ ,\quad k>-1 \quad , \tag{3.21}$$

which we call the generalised (euclidean) wave group. The Mellin transform $\hat{f}$ of a function f is given by

$$\hat{f}(s) := \int_0^\infty t^{s-1} f(t) \mathrm{d}t \tag{3.22}$$

and defines a holomorphic function in a proper domain of definition. Due to the asymtotics (3.19) of $\Theta_{\Delta_D^{\frac{1}{2}}}(t)$ for $t \to 0^+$ the Mellin trasform of $\Theta_{\Delta_D^{\frac{1}{2}}}(t)$ and therefore also of $\Theta_{\Delta_D^{\frac{1}{2}}}(t,k)$ exists at least for $\operatorname{Re} s > 1$ and $k > -1$. Similarly as in [26] p.310 summation and integration can be interchanged and we get

$$\begin{aligned} \zeta_{\Delta_D^{\frac{1}{2}}}(s,k) &= \frac{1}{\Gamma(s)} \widehat{\Theta_{\Delta_D^{\frac{1}{2}}}}(s,k) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{e}^{-kt} \Theta_{\Delta_D^{\frac{1}{2}}}(t) \mathrm{d}t \ , \quad \operatorname{Re} s > 1 \ , \quad \operatorname{Re} k > -1 \quad . \end{aligned} \tag{3.23}$$

Using the trace formula (2.6) for $\Theta_{\Delta_D^{\frac{1}{2}}}(t)$ one can study the analytic continuation of $\zeta_{\Delta_D^{\frac{1}{2}}}(s,k)$ into the complex s-plane. We shall examine the properties of another spectral zeta function (defined in terms of the eigenvalues instead of the wave numbers) in more detail in section 5.

4. The trace of the resolvent

First of all, we want to express the trace of the resolvent $\left(-\Delta_D + k^2\right)^{-1}$ (the term k^2 instead of $-\lambda$ is for convenience) of $-\Delta_D$ in terms of the primitive periodic orbits of the quantum graph Γ (see section 1). Explicitly, the trace of the resolvent is given by

$$\begin{aligned} t(k) := \operatorname{Tr} \left(-\Delta_D + k^2\right)^{-1} &= \sum_{n=1}^\infty \frac{d(n)}{n^2 + k^2} \\ &= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{(nm)^2 + k^2} \ , \quad k \neq \pm\mathrm{i}, \pm 2\mathrm{i}, \pm 3\mathrm{i}, \dots \quad . \end{aligned} \tag{4.1}$$

Using the identity [24] p.471

$$\sum_{m=0}^\infty \frac{1}{m^2 x^2 + y^2} = \frac{1}{2y^2} + \frac{\pi}{2xy} \coth\left(\frac{\pi y}{x}\right) \tag{4.2}$$

and replacing n by $\frac{2\pi}{\mathfrak{l}_{p,n}}$ (the length of the primitive periodic orbit), we obtain from (4.1)

$$t(k) = \sum_{n=1}^\infty \left[-\frac{1}{2k^2} + \frac{\mathfrak{l}_{p,n}}{4k} \coth\left(\frac{k \mathfrak{l}_{p,n}}{2}\right) \right] \ , \quad k \neq \pm\mathrm{i}, \pm 2\mathrm{i}, \pm 3\mathrm{i}, \dots \quad . \tag{4.3}$$

Using the Laurent expansion at $x = 0$ of $\coth x$

$$\coth x = \frac{1}{x} + \sum_{n=1}^\infty \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} \ , \quad 0 < |x| < \pi \quad , \tag{4.4}$$

we get from (4.3) for $t(k)$ (summation can be interchanged due to the absolute convergence of the sums) the power series

$$
\begin{aligned}
t(k) &= \frac{1}{2k^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{2^{2m}B_{2m}}{(2m)!}\frac{(\pi k)^{2m}}{n^{2m}} \\
&= \frac{1}{2k^2}\sum_{m=1}^{\infty}\left[\frac{2^{2m}B_{2m}}{(2m)!}(\pi k)^{2m}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2m}}\right)\right] \\
&= \frac{1}{2k^2}\sum_{m=1}^{\infty}\frac{2^{2m}B_{2m}\zeta(2m)}{(2m)!}(\pi k)^{2m}\ , \quad 0\le |k|<1 \quad .
\end{aligned}
\tag{4.5}
$$

Note that $t(0)=\operatorname{Tr}(-\Delta_D)^{-1}=\sum\limits_{n=1}^{\infty}\frac{d(n)}{n^2}=\zeta^2(2)$.

5. The trace of the heat kernel and a spectral zeta function

For the trace $\Theta_{\Delta_D}(t,k)$ of the generalized heat kernel $\mathrm{e}^{-k^2t}\mathrm{e}^{t\Delta_D}$ we get

$$
\begin{aligned}
\Theta_{\Delta_D}(t,k) &:= \operatorname{Tr}\mathrm{e}^{t\left(\Delta_D-k^2\right)} =: \mathrm{e}^{-k^2t}\Theta_{\Delta_D}(t) = \sum_{n=1}^{\infty} d(n)\mathrm{e}^{-\left(n^2+k^2\right)t} \\
&= \mathrm{e}^{-k^2t}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\mathrm{e}^{-n^2m^2t}\right) \\
&= \frac{1}{2}\mathrm{e}^{-k^2t}\sum_{n=1}^{\infty}\left[\theta_3\left(0,\frac{\mathrm{i}n^2t}{\pi}\right)-1\right]\ , \quad t>0\ , \quad k\in\mathbb{C}\quad ,
\end{aligned}
\tag{5.1}
$$

where we have introduced the elliptic theta function $\theta_3\left(z,\tau\right)$ ([24] p.371). Since for $\theta_3(0,\mathrm{i}x)$ we have the asymptotics ([24] p.371)

$$
\theta_3(0,\mathrm{i}x) = \begin{cases} 1+\mathrm{O}\left(\mathrm{e}^{-\pi x}\right) & ,\quad x\to\infty \\ \frac{1}{\sqrt{x}}+\mathrm{O}\left(\frac{\mathrm{e}^{-\frac{\pi}{x}}}{\sqrt{x}}\right) & ,\quad x\to 0\quad , \end{cases}
\tag{5.2}
$$

we can apply proposition 3.1 identifying $f(x):=\theta_3(0,\mathrm{i}x^2)-1$ and subsequently setting $x:=\sqrt{\frac{t}{\pi}}$. We then obtain from (5.1) and (5.2) ($k\in\mathbb{C}$, fixed)

$$
\Theta_{\Delta_D}(t,k)\sim\frac{1}{2}\mathrm{e}^{-k^2t}\left[-\sqrt{\frac{\pi}{t}}\left(\frac{1}{2}\ln\left(\frac{t}{\pi}\right)-\mathrm{I}_f^{**}\right)+\frac{1}{2}\right]\ , \quad t\to 0^+\quad ,
\tag{5.3}
$$

where we have defined

$$
\mathrm{I}_f^{**} := \int_0^{\infty}\left[\theta_3\left(0,\mathrm{i}t^2\right)-1-\frac{\mathrm{e}^{-t}}{t}\right]\mathrm{d}t\quad .
\tag{5.4}
$$

Using the Taylor expansion of e^x we get for the asymptotics of $\Theta_{\Delta_D}(t,k)$ from (5.3) ($k\in\mathbb{C}$, fixed)

$$
\begin{aligned}
\Theta_{\Delta_D}(t,k) \sim &-\sum_{n=0}^{\infty}\frac{\sqrt{\pi}}{4}\frac{k^{2n}}{n!}t^{n-\frac{1}{2}}\ln t \\
&+\sum_{n=0}^{\infty}\frac{\sqrt{\pi}}{2}\left(\mathrm{I}_f^{**}+\frac{1}{2}\ln\pi\right)\frac{k^{2n}}{n!}t^{n-\frac{1}{2}} \\
&+\sum_{n=0}^{\infty}\frac{k^{2n}}{4n!}t^{n}\ , \quad t\to 0^+\quad .
\end{aligned}
\tag{5.5}
$$

We define analogously to (3.20) the generalised spectral zeta function for $-\Delta_D$

$$\zeta_{\Delta_D}(s,k) := \mathrm{Tr}\left(\left(-\Delta_D + k^2\right)^{-s}\right) = \sum_{n=1}^{\infty} \frac{d(n)}{\left(n^2+k^2\right)^s} \ , \quad \mathrm{Re}\, s > \frac{1}{2} \ , \quad k \neq \pm \mathrm{i}n \ , \quad n \in \mathbb{N} \tag{5.6}$$

(defined by the principal branch of the logharitm) and observe that (see (4.1))

$$\zeta_{\Delta_D}(1,k) = t(k) \ , \quad k \neq \pm \mathrm{i}n \ , \quad n \in \mathbb{N} \tag{5.7}$$

and

$$\begin{aligned} \zeta_{\Delta_D}(s,0) &= \sum_{n=1}^{\infty} \frac{d(n)}{n^{2s}} \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(nm)^{2s}} \\ &= \zeta^2(2s) \ , \quad \mathrm{Re}\, s > \frac{1}{2} \quad . \end{aligned} \tag{5.8}$$

With the same arguments as in section 3 for the expression (3.23) we obtain for $\zeta_{\Delta_D}(s,k)$

$$\begin{aligned} \zeta_{\Delta_D}(s,k) &= \frac{1}{\Gamma(s)} \widehat{\Theta_{\Delta_D}}(s,k) \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \mathrm{e}^{-k^2 t} \Theta_{\Delta_D}(t) \mathrm{d}t \ , \quad \mathrm{Re}\, s > \frac{1}{2} \ , \quad \mathrm{Re}\, k^2 > -1 \quad . \end{aligned} \tag{5.9}$$

We observe that $\frac{1}{\Gamma(s)} : \mathbb{C} \to \mathbb{C}$ is an entire function and has simple zeros at $s_n = -n$, $n \in \mathbb{N}_0$ with respective first Taylor series coefficient (at $s_n = 0, -1, -2, -3, \dots$) $a_1 = (-1)^n n!$, $n \in \mathbb{N}_0$. Furthermore, $\frac{1}{\Gamma(s)}$ possesses at $\frac{1}{2} - n$, $n \in \mathbb{N}$ the Taylor expansion (setting $x = s + \left(n - \frac{1}{2}\right)$, $n \in -\mathbb{N}_0$)

$$\begin{aligned} \frac{1}{\Gamma(s)} &= \frac{1}{\Gamma\left(x - \left(n - \frac{1}{2}\right)\right)} \\ &= \frac{1}{\Gamma\left(\frac{1}{2} - n\right)} - \frac{\psi\left(\frac{1}{2} - n\right)}{\Gamma\left(\frac{1}{2} - n\right)} \left(s + \left(n - \tfrac{1}{2}\right)\right) + \mathrm{O}\left(\left(s + \left(n - \tfrac{1}{2}\right)\right)^2\right) \ , \end{aligned} \tag{5.10}$$

where we have introduced the digamma function $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$. In order to obtain the analytic continuation of $\zeta_{\Delta_D}(s,k)$ as a function of s (for fixed $k \neq \pm \mathrm{i}n$, $n \in \mathbb{N}$), we need the following proposition [26] p.307:

Proposition 5.1. *Let $f(t) : (0,\infty) \to \mathbb{C}$ be a continuous function and of rapid decay at infinity (i.e. $t^a f(t)$ is bounded on (b,∞) for some $b > 0$ and for any $a \in \mathbb{R}$). Furthermore, there exists an asymptotic expansion for $f(t)$, $t \to 0^+$, of the form*

$$f(t) \sim \sum_{j=1}^{\infty} a_j t^{\alpha_j} (\ln t)^{m_j} \ , \quad t \to 0^+ \quad , \tag{5.11}$$

where α_j is an increasing sequence of real numbers tending to ∞ (maybe finitely many are negative) and $m_j \in \mathbb{Z}_{\geq 0}$ are arbitrary. Then the Mellin transform $\hat{f}(s)$ has a meromorphic extension to all of $\mathbb{C}$ with poles at $s = -\alpha_j$ and respective principal part $\left(\mathrm{PP}\left[\hat{f}\right]_{-\alpha_j}\right)(s)$ (the sum gathers all contributions of (5.11) corresponding to the same value of α_j)

$$\left(\mathrm{PP}\left[\hat{f}\right]_{-\alpha_j}\right)(s) = \sum_{m_j} a_j (-1)^{m_j} \frac{(m_j)!}{(\alpha_j + s)^{m_j+1}} \quad . \tag{5.12}$$

Applying proposition (5.1) to the zeta function (5.9) we obtain:

Proposition 5.2. *$\zeta_{\Delta_D}(s,k)$ defined in (5.5) ($k \neq \pm \mathrm{i}n$, $n \in \mathbb{N}$, fixed) has as a function of s a meromorphic extension to all of $\mathbb{C}$ with poles at $s_n = -\left(n - \frac{1}{2}\right)$, $n \in \mathbb{N}_0$, and respective principal part*

$$\begin{aligned}\left(\mathrm{PP}\left[\zeta_{\Delta_D}\right]_{s_n}\right)(s,k) = & \frac{\sqrt{\pi}k^{2n}}{4\Gamma\left(\frac{1}{2}-n\right)n!}\frac{1}{\left(s+\left(n-\frac{1}{2}\right)\right)^2} \\ & + \left(\left(\mathrm{I}_f^{**} + \frac{1}{2}\ln\pi\right)\frac{\sqrt{\pi}k^{2n}}{2\Gamma\left(\frac{1}{2}-n\right)n!} - \frac{\sqrt{\pi}\psi\left(\frac{1}{2}-n\right)}{4\Gamma\left(\frac{1}{2}-n\right)}\frac{k^{2n}}{n!}\right)\frac{1}{s+\left(\frac{1}{2}-n\right)} \quad .\end{aligned} \tag{5.13}$$

Furthermore, the function $\zeta_{\Delta_D}(s,0)$ has "trivial" zeros at $s = -n, \in \mathbb{N}$ (see (5.8)).

A direct consequence of proposition 5.2 is that for $k = 0$ the poles at $s_n = -\left(n - \frac{1}{2}\right)$, $n \in \mathbb{N}$, vanish in agreement with (5.8). Furthermore, with (3.10) we get for $\zeta_{\Delta_D}(s,0)$ at $s = \frac{1}{2}$

$$\zeta_{\Delta_D}(s,0) = \zeta^2(2s) = \frac{1}{4\left(s-\frac{1}{2}\right)^2} + \frac{\gamma}{s-\frac{1}{2}} + \mathrm{O}(1) \quad . \tag{5.14}$$

On the other hand we obtain from (5.13) for $k = 0$ and $n = 0$ with $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2$ ([24] p.3,15)

$$\left(\mathrm{PP}\left[\zeta_{\Delta_D}\right]_{\frac{1}{2}}\right)(s,k) = \frac{1}{4\left(s-\frac{1}{2}\right)^2} + \left[\left(\frac{1}{2}\,\mathrm{I}_f^{**} + \frac{1}{4}\ln\pi\right) + \frac{\gamma + 2\ln 2}{4}\right]\frac{1}{s-\frac{1}{2}} \quad . \tag{5.15}$$

Due to (5.8), we conclude by comparison of (5.14) and (5.15)

$$\begin{aligned}\gamma &\stackrel{!}{=} \frac{1}{2}\,\mathrm{I}_f^{**} + \frac{1}{4}\ln\pi + \frac{\gamma+2\ln 2}{4} \\ \Leftrightarrow \mathrm{I}_f^{**} &\stackrel{!}{=} \frac{3}{2}\gamma - \ln\left(2\sqrt{\pi}\right) \quad .\end{aligned} \tag{5.16}$$

We can compute $\zeta_{\Delta_D}(s,k)$ from the trace formula (2.1) using the function $f_k(s,p) := \frac{1}{(p^2+k^2)^s}$, $\mathrm{Re}\, s > \frac{1}{2}$, $k \neq \pm \mathrm{i}n$, $n \in \mathbb{N}$ which also fulfills as a funciton of p the requirements of theorem 2.1. Due to the identity (2.6) we split $\zeta_{\Delta_D}(s,k)$

$$\zeta_{\Delta_D}(s,k) := \zeta^W_{\Delta_D}(k,k) + \zeta^{Osc.}_{\Delta_D}(s,k) \tag{5.17}$$

analogously to (3.14) into a Weyl term

$$\zeta^W_{\Delta_D}(s,k)(t) := \int_0^\infty (\ln p + 2\gamma) f_k(s,p)\mathrm{d}p + \frac{f_k(s,0)}{4} \ , \quad \mathrm{Re}\, s > \frac{1}{2} \ , \quad k \neq \pm \mathrm{i}n \ , \quad n \in \mathbb{N} \quad , \tag{5.18}$$

and an "oscillatory" term

$$\zeta^{Osc.}_{\Delta_D}(s,k) := 2\pi \sum_{n=1}^\infty \int_0^\infty \left[\frac{2}{\pi}K_0\left(4\pi\sqrt{np}\right) - Y_0\left(4\pi\sqrt{np}\right)\right] f_k(s,p)\mathrm{d}p \quad . \tag{5.19}$$

In order to evaluate $\zeta^W_{\Delta_D}(s,k)$ we calculate the integrals of each summand in (5.18) separately. We obtain for the integral in the first summand ([31] p.527)

(5.20)

$$\begin{aligned}\int_0^\infty \frac{\ln p}{(p^2+k^2)^s}\mathrm{d}p &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{e}^{-k^2 t}\left[\int_0^\infty \mathrm{e}^{-p^2 t}\ln p\,\mathrm{d}p\right]\mathrm{d}t \\ &= \frac{\Gamma\left(\frac{1}{2}\right)}{4\Gamma(s)}\int_0^\infty t^{s-\frac{3}{2}}\mathrm{e}^{-k^2 t}\left[\psi\left(\frac{1}{2}\right) - \ln t\right]\mathrm{d}t \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{4\Gamma(s)k^{2s-1}}\left[\psi\left(\frac{1}{2}\right) - \psi\left(s-\frac{1}{2}\right) + 2\ln k\right] \ , \quad \mathrm{Re}\, s > \frac{1}{2} \ , \quad \mathrm{Re}\, k^2 > 0 \quad ,\end{aligned}$$

and for the second summand with a similar calculation ([31] p.527)

$$\int_0^\infty \frac{2\gamma}{(p^2+k^2)^s}\,\mathrm{d}p = \frac{2\gamma\Gamma\left(\frac{1}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{2\Gamma(s)k^{2s-1}}\ , \qquad \operatorname{Re} s > \frac{1}{2}\ , \quad \operatorname{Re} k^2 > 0 \quad . \tag{5.21}$$

Therefore, we get for $\zeta^W_{\Delta_D}(s,k)$ (using the identities $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $\psi\left(\frac{1}{2}\right)=-\gamma-2\ln 2$ ([24] p.3,15))

$$\zeta^W_{\Delta_D}(s,k) = \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{2\Gamma(s)k^{2s-1}}\left[2\gamma+\ln k - \frac{\psi\left(s-\frac{1}{2}\right)+\gamma+2\ln 2}{2}\right] + \frac{1}{4k^{2s}}\ , \qquad \operatorname{Re} s > \frac{1}{2}\ , \quad \operatorname{Re} k^2 > 0 \quad . \tag{5.22}$$

Of course, with the representation (5.22) the function $\zeta^W_{\Delta_D}(s,k)$ can as a function of s ($\operatorname{Re} k^2 > 0$, fixed) be analytically continued to $\mathbb{C}\setminus\left\{s_n=-\left(n-\frac{1}{2}\right);\quad n\in\mathbb{N}_0\right\}$. Using the Taylor expansion (5.10) at $s_0=\frac{1}{2}$ and ([24] p.3,p.13)

$$\begin{aligned} \psi\left(s-\frac{1}{2}\right) &= -\frac{1}{s-\frac{1}{2}} - \gamma + \mathrm{O}\left(s-\frac{1}{2}\right) \quad , \\ \Gamma\left(s-\frac{1}{2}\right) &= \frac{1}{s-\frac{1}{2}} - \gamma + \mathrm{O}\left(s-\frac{1}{2}\right) \quad , \end{aligned} \tag{5.23}$$

we obtain from (5.22) for the principal part of $\zeta^W_{\Delta_D}(s,k)$ at $s_0=\frac{1}{2}$ ($\operatorname{Re} k^2>0$, fixed)

$$\mathrm{PP}\left[\zeta^W_{\Delta_D}\right]_{\frac{1}{2}}(s,k) = \frac{1}{4\left(s-\frac{1}{2}\right)^2} + \frac{\gamma}{s-\frac{1}{2}} + \mathrm{O}(1)\ , \quad s\to 0 \quad , \tag{5.24}$$

in complete agreement with (5.14) (see (5.8)).

We want to investigate the holomorphy of $\zeta_{\Delta_D}(s,k)$ defined in (5.6) as a function on $\mathbb{C}^2$ in s and k. For this reason, we need in the sequel Hartogs' theorem [32] (this version is stated and improved in [33]).

Theorem 5.3 (Hartogs' theorem)**.** *Let Z be a domain in the space of the complex variables $z=(z_1,\dots,z_n)$, and let $W_0\subset W$ be domains in the space of the complex variable $w=(w_1,\dots,w_m)$. If $f(z,w)$ is analytic with respect to the set of variables in the domain $z\in Z$, $w\in W_0$, and if for each fixed $z\in Z$ it is analytic with respect to w in the domain W, then $f(z,w)$ is analytic with respect to the set of variables z, w in the domain $Z\times W$.*

Due to the normal convergence of the series in (5.6) the function $\zeta_{\Delta_D}(s,k)$ ($\operatorname{Re} s>\frac{1}{2}$, fixed) is as a function of k holomorphic on $\mathbb{C}^{\,\mathrm{i}}_{-\mathrm{i}}$, where we have defined

$$\mathbb{C}^{\,\mathrm{i}}_{-\mathrm{i}} := \mathbb{C}\setminus\{z\in\mathbb{C};\quad |\operatorname{Im} z|\geq 1\ ,\quad \operatorname{Re} z = 0\} \quad . \tag{5.25}$$

On the other hand we know from proposition 5.2 that $\zeta_{\Delta_D}(s,k)$ ($k\neq\pm\mathrm{i}n$, $n\in\mathbb{N}$, fixed) is as a function of s holomorphic on $\mathbb{C}\setminus\left\{z_n=-\left(n-\frac{1}{2}\right);\quad n\in\mathbb{N}_0\right\}$. Therefore, we infer ([34] p.135)

Proposition 5.4. *$\zeta_{\Delta_D}(s,k)$ is a holomorphic (and therefore an analytic) function on*

$$Z\times W_0 := \left\{(s,k)\in\mathbb{C}\times\mathbb{C};\quad k\in\mathbb{C}^{\,\mathrm{i}}_{-\mathrm{i}}\ ,\quad s\in\left\{z\in\mathbb{C};\quad \operatorname{Re} s>\frac{1}{2}\right\}\right\} \quad . \tag{5.26}$$

Thus, by the above results for $\zeta_{\Delta_D}(s,k)$ we can apply theorem 5.3 and get

Corollary 5.5. *$\zeta_{\Delta_D}(s,k)$ is a holomorphic (and therefore an analytic) function on*

$$Z\times W := \left\{(s,k)\in\mathbb{C}\times\mathbb{C};\quad k\in\mathbb{C}^{\,\mathrm{i}}_{-\mathrm{i}}\ ,\quad s\in\mathbb{C}\setminus\left\{-\left(n-\frac{1}{2}\right);\quad n\in\mathbb{N}_0\right\}\right\} \quad . \tag{5.27}$$

6. A "secular equation"

For our simple infinite quantum graph Γ the eigenvalues respectively wave numbers are explicitly known. However, in order to investigate the eigenvalues of a more complex infinite quantum graph it would we interesting to find a "secular equation" determining the eigenvalues for the quantum mechanical system $(-\Delta, \mathrm{D}(\Delta))$ (see (1.3) and 1.4). We recall that for a compact quantum graph with E edges, $0 < E < \infty$, this was i.a. done e.g. in [35, 6]. In the setting of Kostrykin and Schrader [6] the secular equation $F_E(k)$ is a function defined by a determinant

$$F_E(k) := \det\left(\mathbb{1}_{2E\times 2E} - S(k)_{2E} T_{2E}(k)\right) \;, \quad k \in \mathbb{R} \quad , \tag{6.1}$$

and the nonzero real zeros k_n of $F_E(k)$ are exactly the wave numbers (and the orders of the zeros are exactly the multiplicities of the corresponding eigenvalues) of the corresponding quantum graph. The matrix S is called the S-matrix and describes the quantum mechanical scattering at the vertices (see [3, 5] for a detailed discussion). The matrix T_{2E} can be interpreted as a "transfer" matrix [35] describing the propagation of the eigenfunctions along the edges. It has in the setting of [6] the form

$$T_{2E}(k) := \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}k\boldsymbol{l}} \\ \mathrm{e}^{\mathrm{i}k\boldsymbol{l}} & 0 \end{pmatrix} \;, \quad \left(\mathrm{e}^{\mathrm{i}k\boldsymbol{l}}\right)_{mn} := \delta_{mn}\mathrm{e}^{\mathrm{i}kl_m} \;, \quad 1 \le m, n \le E \quad , \tag{6.2}$$

where $0 < l_m < \infty$ denotes the length of the edge e_m. Clearly, the definition (6.2) doesn't make sense for an infinite quantum graph. Therefore, we make a rearrangement of the matrix entries and define

$$\widetilde{T}_{2E}(k) := \begin{pmatrix} T^1(k) & & 0 \\ & \ddots & \\ 0 & & T^E(k) \end{pmatrix} \;, \quad T^n(k) := \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}kl_n} \\ \mathrm{e}^{\mathrm{i}kl_n} & 0 \end{pmatrix} \;, \quad 1 \le n \le E \quad . \tag{6.3}$$

We remark that the rearrangement can be achieved by a permutation matrix P by $\widetilde{T}_{2E}(k) = P^+ T_{2E}(k) P$. The S-matrix is for a compact quantum graph with E edges for Dirichlet boundary conditions ("hard wall" reflection) the unit matrix $S_{2E}(k) = \mathbb{1}_{2E\times 2E}$ and thus $\widetilde{S}_{2E}(k) := P^+ S_{2E}(k) P = \mathbb{1}_{2E\times 2E}$. Therefore, the secular equation in (6.1) is equivalent to (compact graph)

$$\widetilde{F}_E^D(k) := \det\left(\mathbb{1}_{2E\times 2E} - \widetilde{T}_{2E}(k)\right) \quad . \tag{6.4}$$

We now extend the rearrangement to our infinite quantum graph (see section 1) and observe that the eigenvalue problem $\widetilde{T}_\infty(k)x = x$ with $x \in l^2$ ($\widetilde{T}_\infty(k)$ as a linear and bounded operator on the Hilbert space of square summable sequences) yields exactly the wave numbers $k_{n,m} = nm$. However, if we take over the definition of $\widetilde{F}_E^D(k)$ in (6.4) for our infinite quantum graph with Dirichlet boundary conditions there arises a problem since

$$\widetilde{F}_\infty^D(k) := \text{“}\det\left(\mathbb{1} - \widetilde{T}_\infty(k)\right)\text{”} := \prod_{m=1}^{\infty}\left(1 - \mathrm{e}^{\frac{2\pi \mathrm{i}k}{m}}\right) = 0 \;, \quad \text{for all} \quad k \in \mathbb{C} \quad . \tag{6.5}$$

Therefore, we perform a proper "renormalization" of $\widetilde{F}_\infty^D(k)$ in such a way that the proper "renormalized" function $\widetilde{F}_\infty^{D,R}(k)$ is zero at precisely the wave numbers $\pm\mathrm{i}k_{n,m}$ of our quantum graph Γ. We remark that the operator $\widetilde{T}_\infty(k)$ is not an element of any trace ideal T_p, $p \le 1$ and therefore we cannot apply results of [36] for renormalization. Therefore, in order to find a proper "renormalized" function we use a technique of quantum field theory which was mathematically founded by [37] and first applied to physics (quantum field theory) by [38]. But at first we remark that the function

$$Z(k) := \frac{1}{2\pi}\prod_{n=1}^{\infty}\left(1 + \frac{k^2}{n^2}\right)^{d(n)} \;, \quad k \in \mathbb{C} \quad , \tag{6.6}$$

defines a holomorphic (and therefore an analytic) function on $\mathbb{C}$ since the product in (6.6) is normally convergent on $\mathbb{C}$. Furthermore, it is obvious from (6.6) that the zeros z_l, $l = 1, \ldots, \infty$ of $Z(k)$ are exactly located at the wave numbers $\pm\mathrm{i}k_{n,m}$, $n, m = 1\ldots, \infty$ (identical with $\pm\mathrm{i}\mathbb{N}$) of our quantum graph Γ and the order of the zero $z_l = \pm\mathrm{i}l$ is exactly the multiplicity $d\left(k_l\right)$ of the corresponding wave number $k_l = l$.

Now, we define the functional determinant by [37, 38] (see (5.25))

$$\mathfrak{Z}(k) := \det\left(-\Delta_D + k^2\right) := \exp\left(-\frac{\partial\zeta_{\Delta_D}(s,k)}{\partial s}\bigg|_{s=0}\right) \ , \quad k \in \mathbb{C}_{-\mathrm{i}}^{\ \mathrm{i}} \quad . \tag{6.7}$$

Due to corollary 5.5 the function $\mathfrak{Z}(k)$ is well defined and is a holomorphic (and therefore an analytic) function on $\mathbb{C}_{-\mathrm{i}}^{\ \mathrm{i}}$. Furthermore, we define an Euler product for our quantum graph (product over the primitive periodic orbit lengths $\mathfrak{l}_{p,n} = \frac{2\pi}{n}$)

$$\mathcal{Z}(k) := \prod_{n=1}^{\infty}\left[\frac{\exp\left(\frac{k\mathfrak{l}_{p,n}}{2}\right)}{k\mathfrak{l}_{p,n}}\left(1-\exp\left(-k\mathfrak{l}_{p,n}\right)\right)\right] = \prod_{n=1}^{\infty}\left(\frac{\sinh\left(\frac{k\mathfrak{l}_{p,n}}{2}\right)}{\left(\frac{k\mathfrak{l}_{p,n}}{2}\right)}\right) \ , \quad k \in \mathbb{C} \quad . \tag{6.8}$$

We remark that the function in (6.8) is well defined since $\sinh x = x + \frac{1}{6}x^3 + \mathrm{O}(x^5)$ for $x \to 0$ (0 is a removable singularity). Furthermore, $\mathcal{Z}(k)$ is a holomorphic (and therefore an analytic) function on $\mathbb{C}$ since the product is compact convergent.

Theorem 6.1. *$\mathfrak{Z}(k)$ can be analytically continued on $\mathbb{C}$ as an entire function. Furthermore, the following identity holds*

$$\mathfrak{Z}(k) = \mathcal{Z}(k) = Z(k) \ , \quad \textit{for all} \quad k \in \mathbb{C} \quad . \tag{6.9}$$

Proof. In order to show $\mathfrak{Z}(k) = \mathcal{Z}(k)$ our strategy is to prove this in a neighbourhood $U_\delta(0)$, $\delta > 0$ at $k = 0$ and then by analytical continuation to conclude (6.9) on $\mathbb{C}$. Therefore, we calculate (see (5.6) [principal branch of the logarithm])

$$-\frac{\partial\zeta_{\Delta_D}(s,k)}{\partial s} = -\sum_{n=1}^{\infty} d(n)\frac{\ln\left(k^2+n^2\right)}{\left(k^2+n^2\right)^s} \ , \quad \operatorname{Re} s > \frac{1}{2} \ , \quad k \in \mathbb{C}_{-\mathrm{i}}^{\ \mathrm{i}} \quad . \tag{6.10}$$

Furthermore, we get

$$\begin{aligned} -\frac{\partial^2\zeta_{\Delta_D}(s,k)}{\partial k\partial s} =& -2k\sum_{n=1}^{\infty}\frac{d(n)}{\left(k^2+n^2\right)^{s+1}} \\ &+ 2sk\sum_{n=1}^{\infty} d(n)\frac{\ln\left(k^2+n^2\right)}{\left(k^2+n^2\right)^{s+1}} \ , \quad \operatorname{Re} s > -\frac{1}{2} \ , \quad k \in \mathbb{C}_{-\mathrm{i}}^{\ \mathrm{i}} \quad , \end{aligned} \tag{6.11}$$

where in the last line we have used the uniqueness of the analytic continuation. Setting $s = 0$ (see (6.11) and (4.1)) we obtain

$$\frac{\partial}{\partial k}\left[\frac{\partial\zeta_{\Delta_D}(s,k)}{\partial s}\bigg|_{s=0}\right] = 2k\sum_{n=1}^{\infty}\frac{d(n)}{k^2+n^2} = 2kt(k) \ , \quad |k| < 1 \quad . \tag{6.12}$$

On the other hand we have (see (6.6) and (4.1))

$$\frac{\mathrm{d}}{\mathrm{d}k}\ln Z(k) = 2k\sum_{n=1}^{\infty}\frac{d(n)}{k^2+n^2} = 2kt(k) = \frac{\partial}{\partial k}\left[\frac{\partial\zeta_{\Delta_D}(s,k)}{\partial s}\bigg|_{s=0}\right] \ , \quad |k| < 1 \quad . \tag{6.13}$$

By (5.8) and [39][p.807] ($\zeta'(0) = -\frac{1}{2}\ln(2\pi)$, $\zeta(0) = -\frac{1}{2}$) we get

$$\exp\left(-\frac{\partial\zeta_{\Delta_D}(s,0)}{\partial s}\bigg|_{s=0}\right) = \exp\left(-4\zeta'(0)\zeta(0)\right) = \frac{1}{2\pi} = Z(0) = \mathfrak{Z}(0) := \det\left(-\Delta_D\right) \quad . \tag{6.14}$$

The identity $\mathcal{Z}(k) = Z(k)$ can be also shown by calculating again the logarithmic derivative $\frac{\mathrm{d}}{\mathrm{d}k}\ln\left(\mathcal{Z}(k)\right) = \frac{\mathcal{Z}'(k)}{\mathcal{Z}(k)}$, $k \in \mathbb{C}$, and then comparing with (6.13) but we omit this since this is a straightforward calculation. □

Now we can propose a possible "renormalization" of our original function $\widetilde{F}^D_\infty(k)$ in (6.5). For this purpose we define a truncation operation for our infinite matrices $\mathbb{1}$ and $\widetilde{T}_\infty(k)$ in (6.5) (see (6.3)) and for the function (6.5) (see(6.4)) by

$$\begin{aligned} &\mathbb{1}|_N := \mathbb{1}_{2N} \ , \quad \widetilde{T}_\infty\Big|_N(k) := \widetilde{T}_{2N}(k) \ , \\ &\widetilde{F}^D_\infty\Big|_N(k) := \widetilde{F}_N(k) := \det\left(\mathbb{1}|_N - \widetilde{T}_\infty(k)\Big|_N\right) = \prod_{n=1}^{2N}\left(1 - \mathrm{e}^{\frac{2\pi\mathrm{i}k}{n}}\right) \ , \quad N \in \mathbb{N} \quad . \end{aligned} \tag{6.15}$$

We remark that these truncation operations correspond in our case to a truncation of the quantum graph which means that the truncated matrices respectively the truncated function describe a finite quantum graph with N edges which have the lengths $l_n = \frac{\pi}{n}$ and are equipped with Dirichlet boundary conditions. Since as previously mentioned the zeros z_l, $l = 1, \dots, \infty$ of $Z(k)$ are exactly located at the wave numbers $\pm \mathrm{i}k_{n,m}$ and the order of the zeros and the multiplicities of the wave numbers coincide, we define due to theorem 6.1 for the "renormalized" function $\widetilde{F}^{D,R}_\infty(k)$ (see (6.8), $\mathfrak{l}_{p,n} = \frac{2\pi}{n}$)

$$\widetilde{F}^{D,R}_\infty(k) := \lim_{N\to\infty}\left(\frac{(2N)!\mathrm{e}^{k\mathcal{L}_{2N}}}{(2\pi k)^{2N}}\ \widetilde{F}^D_\infty\Big|_N(\mathrm{i}k)\right) = \mathcal{Z}(k) \ , \quad k \in \mathbb{C} \quad , \tag{6.16}$$

where $\mathcal{L}_{2N} := \sum\limits_{n=1}^{2N} l_n$ is the total length of the truncated quantum graph consisting of N edges. Due to (6.15) this can be regarded as a "renormalized" limit of the secular equation (6.4) for finite quantum graphs with Dirichlet boundary conditions.

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Sebastian Endres and Frank Steiner, Institut für Theoretische Physik, Universität Ulm, Albert-Einstein-Allee 11, 89081 Ulm, Germany

E-mail address: frank.steiner@uni-ulm.de
E-mail address: sebastian.endres@uni-ulm.de