

# Gauges and Cosmological Backreaction

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We present a formalism for spatial averaging in cosmology applicable to general spacetimes and coordinates, and allowing the easy incorporation of a wide variety of matter sources. We apply this formalism to a Friedmann-LeMaître-Robertson-Walker universe perturbed to second-order and present the corrections to the background in an unfixed gauge. We then present the corrections that arise in uniform curvature and conformal Newtonian gauges.

## I. INTRODUCTION

The issue of the averaging, or fitting, problem in cosmology is an old one and addresses the question of whether a Friedmann-LeMaître-Robertson-Walker (FLRW) metric is a reasonable approximation to the metric of the universe on large scales, and whether the evolution of the universe follows the usual FLRW dynamics. As the Einstein tensor is nonlinear, even should the metric of the universe be of FLRW form there is no guarantee that the same is true for the dynamics, and corrections therefore arise that act as an effective fluid. This effect is typically known as cosmological backreaction. The study of averaging in cosmology dates back at least to the 1960s [1] and was revisited in subsequent decades (see for example [2, 3, 4, 5, 6, 7] for an inexhaustive selection of studies). Following the discovery of the “dark energy” in the late 1990s there was a great increase of study in this area (e.g. [8, 9, 10, 11, 12]). Recent studies have begun to quantify the effect with varying results; see for example [13, 14, 15, 16, 17] and for further references see [18]. However, many of these studies were limited to pure dust (“Einstein-de Sitter”) universes, and frequently to spatial slicings in synchronous coordinates. In previous work [19] (hereafter Paper I) and [18] (hereafter Paper II), as in for example [20, 21], a Newtonian slicing was employed instead. Paper II additionally introduced a general formalism applicable to systems with a variety of fluids for a wide range of redshifts.

When considering the issue of backreaction in perturbation theory, the question of gauges has to be carefully considered. The results of, for example, [13] and Paper II cannot be directly compared, as the former were calculated in conformal synchronous gauge and the latter in Newtonian gauge. It is not immediately obvious that there does not exist a gauge that is particularly suited to studies of backreaction – in which the correction terms take their simplest forms or, indeed, could be shown to vanish identically. As yet it has been impossible to study this issue, as to consider general gauges it is necessary to possess a formalism that incorporates a shift vector. The central aim of this paper is to develop this formalism, which is also necessary to consider general inhomogeneous systems, and to apply it to perturbed Robertson-Walker universes in an unfixed gauge. It is then a simple matter to choose two promising gauges – the uniform curvature and the conformal Newtonian gauges – and reduce the general expressions to particular cases. We find that the backreaction cannot be removed through a judicious choice of gauge and that the two most obvious simple choices remain relatively complicated to consider. In §II we present a brief overview of the 3+1 separation before generalising the familiar spatial averaging procedure to the most general case. We apply this general formalism to a linearly-perturbed FLRW universe in an unfixed gauge in §III A and present the backreaction quantities in full detail before considering the specialisation to flat and conformal Newtonian gauges in §III B. We end with a brief discussion.

## II. GENERALISED AVERAGING IN COSMOLOGY

The standard spatial averaging procedure in cosmology is based on a 3+1 split, where spacetime is foliated with purely spatial hypersurfaces upon which an averaging domain is defined. Assuming global hyperbolicity and a metric

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signature  $(-+++)$ , a spacetime can be foliated with a family of 3-surfaces described by coordinates  $x^\mu = (t, x^a)$  and with a normal 4-vector

$$n^\mu = \frac{1}{\alpha}(1, -\beta^i), \quad n^\mu n_\mu = -1. \quad (1)$$

Here the threading between corresponding events on neighbouring surfaces is defined by the lapse function  $\alpha$  and the shift vector  $\beta^i$ . The projection tensor onto the 3-surfaces is

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad (2)$$

and the induced 3-metric is

$$h_{ij} = g_{\mu\nu} h_i^\mu h_j^\nu. \quad (3)$$

The line element in these coordinates is then

$$ds^2 = (-\alpha^2 + \beta_i \beta^i) dt^2 + 2\beta_i dt dx^i + h_{ij} dx^i dx^j, \quad (4)$$

where  $\beta_i = h_{ij} \beta^j$ .

The embedding of the hypersurfaces is described by the extrinsic curvature, which can be written as the Lie derivative of the metric along the normal,

$$K_{ij} = -\frac{1}{2} \mathcal{L}_n h_{ij}. \quad (5)$$

This can be rewritten as

$$\alpha \mathcal{L}_n h_{ij} = \frac{\partial}{\partial t} h_{ij} - 2\mathcal{D}_{(i} \beta_{j)} = -2\alpha K_{ij} \quad (6)$$

where  $\mathcal{D}_i$  is the covariant derivative on the 3-surface and brackets denote symmetrisation on the enclosed indices.

Using  $\varrho$ ,  $J_i$  and  $S_{\mu\nu}$  to denote a fluid's energy density, current density and stress tensor on the 3-surface respectively, the 3+1 decomposition of the stress-energy tensor is

$$T_{\mu\nu} = \varrho n_\mu n_\nu + 2n_{(\mu} J_{\nu)} + S_{\mu\nu} \iff \varrho = n^\mu n^\nu T_{\mu\nu}, \quad J_i = -n^\mu T_{i\mu}, \quad S_{\mu\nu} = h_i^\mu h_j^\nu T_{\mu\nu}. \quad (7)$$

We retain  $\rho$  and  $p$  to denote the fluid *rest-frame* energy density and isotropic pressure respectively. Projecting the Einstein equations onto the hypersurface and along its normal results in the Hamilton constraint equation

$$\mathcal{R} + K^2 - K_j^i K_i^j = 16\pi G \varrho + 2\Lambda, \quad (8)$$

where  $\mathcal{R}$  is the Ricci scalar on the surface and  $K = K_i^i$ , and the evolution equation

$$\begin{aligned} \alpha \frac{\partial}{\partial t} K_{ij} &= \alpha^2 (\mathcal{R}_{ij} - 2K_i^a K_{aj} + K K_{ij} - 8\pi G S_{ij} + 4\pi G h_{ij} (S - \varrho) - \Lambda h_{ij}) \\ &\quad - \alpha (\mathcal{D}_i \mathcal{D}_j \alpha - \beta^\alpha \mathcal{D}_\alpha K_{ij} - 2K_{a(i} \mathcal{D}_{j)} \beta^a). \end{aligned} \quad (9)$$

It also results in a momentum constraint equation with which we are not here concerned. For more details on the 3+1 split, we refer the reader to [22, 23, 24].

The coordinate freedoms now allow us to eliminate four degrees of freedom. In previous studies of cosmological backreaction it has been standard to choose a synchronous slicing by setting  $\alpha = 1$ ,  $\beta^i = 0$  (e.g. [8, 10, 13]). An alternative has been to employ a ‘‘Newtonian’’ slicing by choosing  $\beta^i = 0$  and diagonalising the scalar sector of  $h_{ij}$  (e.g. [18, 19, 20, 21]). However, a general metric contains a shift and while we can always choose a coordinate system in which this vanishes, such is not necessarily the most natural or convenient. Likewise, for work in perturbation theory we can consider a variety of different gauges and to consider these and the transformations between them it is most convenient to retain the shift vector. In this manner a transformation between, for example, a perturbed FLRW universe in Newtonian gauge and a perturbed FLRW universe in uniform-curvature gauge can be easily effected. In the following we do not fix our slicing and the coordinate freedoms are left implicit.

Now let

$$V_{\mathcal{D}} = \int_{\mathcal{D}} \sqrt{h} d^3 \mathbf{x} \quad (10)$$

be the definition of the volume of a finite domain  $\mathcal{D}$  lying on the inhomogeneous 3-surface where  $h$  is the determinant of the 3-metric. Then the average of a scalar quantity  $A$  in this domain can be defined as

$$\langle A \rangle = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} A \sqrt{h} d^3 \mathbf{x}. \quad (11)$$

This averaging procedure cannot be sensibly applied to tensorial quantities since it is not invariant under a change of coordinates; for a scalar quantity, however, it retains its meaning. This averaging procedure has been used by most of the studies into cosmological averaging (e.g. [9, 11, 12]); for a brief overview of alternatives see [25, 26]. We can define an average ‘‘Hubble rate’’ from the derivative of the volume with respect to time,

$$\frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} \left( \frac{1}{2} h^{ij} \dot{h}_{ij} \right) \sqrt{h} d^3 \mathbf{x} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} (-\alpha K + \mathcal{D}^i \beta_i) \sqrt{h} d^3 \mathbf{x}$$

by

$$3 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} = -\langle \alpha K + \mathcal{D}^i \beta_i \rangle \quad (12)$$

with  $\mathcal{D}^i = h^{ij} \mathcal{D}_j$ .<sup>1</sup> Combining this with the averaged Hamiltonian constraint yields

$$\left( \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 = \frac{8\pi G}{3} \langle \alpha^2 \varrho \rangle + \frac{1}{3} \langle \alpha^2 \rangle \Lambda - \frac{1}{6} (\mathcal{R}_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}^T) \quad (13)$$

which can be interpreted as a Friedmann equation for  $a_{\mathcal{D}}$  in the domain, where

$$\mathcal{R}_{\mathcal{D}} = \langle \alpha^2 \mathcal{R} \rangle, \quad \mathcal{Q}_{\mathcal{D}}^T = \left\langle \alpha^2 \left( K^2 - K_j^i K_i^j \right) \right\rangle - \frac{2}{3} \langle \alpha K - \mathcal{D}^i \beta_i \rangle^2 \quad (14)$$

are corrections from the averaged spatial curvature and the kinematical backreaction. The latter can be separated into the usual  $\mathcal{Q}_{\mathcal{D}}$  (e.g. [8, 9, 10] and Paper II) which is related to the variance of the extrinsic curvature, and  $\beta_{\mathcal{D}}^{\mathcal{Q}}$  which is dependent on the divergence of the shift,

$$\mathcal{Q}_{\mathcal{D}} = \left\langle \alpha^2 \left( K^2 - K_j^i K_i^j \right) \right\rangle - \frac{2}{3} \langle \alpha K \rangle^2, \quad \beta_{\mathcal{D}}^{\mathcal{Q}} = \frac{2}{3} \langle \mathcal{D}^i \beta_i \rangle (2 \langle \alpha K \rangle - \langle \mathcal{D}^i \beta_i \rangle). \quad (15)$$

Spatial averaging does not commute with differentiation with respect to time. The commutator between these operations applied to a scalar quantity  $A$  is given by

$$\left\langle \dot{A} \right\rangle = \langle \dot{A} \rangle + \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} \langle A \rangle - \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} \left( \frac{1}{2} A h^{ij} \dot{h}_{ij} \right) \sqrt{h} d^3 \mathbf{x} \quad (16)$$

which can be written

$$\left\langle \dot{A} \right\rangle = \langle \dot{A} \rangle + \langle A (\alpha K - \mathcal{D}^i \beta_i) \rangle - \langle A \rangle \langle \alpha K - \mathcal{D}^i \beta_i \rangle. \quad (17)$$

The trace of the evolution equation for the extrinsic curvature is

$$\alpha \dot{K} = \alpha^2 (\mathcal{R} + K^2 - 12\pi G \varrho + 4\pi G S - 3\Lambda) - \alpha \mathcal{D}^i \mathcal{D}_i \alpha + \alpha \beta^a \mathcal{D}_a K. \quad (18)$$

Since

$$\left\langle \alpha \dot{K} \right\rangle = \langle \alpha K \rangle + \langle \alpha^2 K^2 \rangle - \langle \alpha K \rangle^2 + \langle \alpha K \rangle \langle \mathcal{D}^i \beta_i \rangle - \langle \alpha K \mathcal{D}^i \beta_i \rangle - \langle \dot{\alpha} K \rangle \quad (19)$$

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<sup>1</sup> The definition of the Hubble rate is not unique and it is common to instead define it from the expansion scalar of a fluid. The relation between the approaches was recently calculated in [27]. We retain our present definition to avoid connecting our slicing directly with fluid quantities while leaving free the possibility of doing so later. This simplifies the presentation at the cost of physical clarity.

and

$$\langle \alpha K \rangle \cdot = -3 \frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 3 \left( \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 + \langle \mathcal{D}^i \beta_i \rangle \cdot, \quad (20)$$

we have after some manipulation and the use of (13) that

$$\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -\frac{4\pi G}{3} \langle \alpha^2 (\varrho + S) \rangle + \frac{1}{3} \langle \alpha^2 \rangle \Lambda + \frac{1}{3} (\mathcal{Q}_{\mathcal{D}}^T + \mathcal{P}_{\mathcal{D}} + \beta_{\mathcal{D}}^P) \quad (21)$$

where the standard dynamical backreaction [18, 19, 28] is

$$\mathcal{P}_{\mathcal{D}} = \langle \alpha \mathcal{D}^i \mathcal{D}_i \alpha \rangle - \langle \dot{\alpha} K \rangle, \quad (22)$$

and the component arising from the variation of the shift is

$$\beta_{\mathcal{D}}^P = \left\langle (\mathcal{D}^i \beta_i) \cdot + (\mathcal{D}^i \beta_i)^2 - 2\alpha K \mathcal{D}^i \beta_i - \alpha \beta^i \mathcal{D}_i K \right\rangle. \quad (23)$$

Equation (21) can be interpreted as a ‘‘Raychaudhuri equation’’ for the scale factor  $a_{\mathcal{D}}$ .

The energy density and isotropic pressure are defined on the 3-surface in (7). However, it is usual in cosmology to separate the stress-energy tensor with the fluid’s 4-velocity  $u^\mu$  by

$$T_{\mu\nu} = \rho u_\mu u_\nu + p \tilde{P}_{\mu\nu} + 2q_{(\mu} u_{\nu)} + \pi_{\mu\nu} \iff \rho = u^\mu u^\nu T_{\mu\nu}, \quad 3p = \tilde{P}^{\mu\nu} T_{\mu\nu} \quad (24)$$

where

$$\tilde{P}_\nu^\mu = \delta_\nu^\mu + u^\mu u_\nu \quad (25)$$

projects tensors onto the 3-surface orthogonal to the 4-velocity,  $u^\mu u_\mu = -1$  and  $\rho$  and  $p$  are then the fluid’s rest-frame energy density and isotropic pressure respectively. With

$$\mathcal{F}_{\mathcal{D}} = \frac{8\pi G}{3} \left\langle (h^{\mu\nu} - \tilde{P}^{\mu\nu}) T_{\mu\nu} \right\rangle = \frac{8\pi G}{3} \langle (n^\mu n^\nu - u^\mu u^\nu) T_{\mu\nu} \rangle \quad (26)$$

a correction defining the difference between the 3-surface and the fluid rest-frames, the effective Friedmann and Raychaudhuri equations can then be written

$$\left( \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 = \frac{8\pi G}{3} \langle \alpha^2 \rho \rangle + \frac{1}{3} \langle \alpha^2 \rangle \Lambda - \frac{1}{6} (\mathcal{R}_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}^T - 6\mathcal{F}_{\mathcal{D}}), \quad (27)$$

$$\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -\frac{4\pi G}{3} \langle \alpha^2 (\rho + 3p) \rangle + \frac{1}{3} \langle \alpha^2 \rangle \Lambda + \frac{1}{3} (\mathcal{P}_{\mathcal{D}} + \beta_{\mathcal{D}}^P + \mathcal{Q}_{\mathcal{D}}^T - 3\mathcal{F}_{\mathcal{D}}). \quad (28)$$

We therefore now have definitions of effective, averaged Friedmann and Raychaudhuri equations valid for any spacetime that allows a 3+1 split, with arbitrary lapse and shift, and an arbitrary number of fluids. Equations (27, 28) are the main results of this work.<sup>2</sup>

It is tempting to immediately define an effective energy density, pressure and equation of state of the ‘‘backreaction fluid’’ and such can certainly be done. Simply interpreting  $\langle \alpha^2 \rho \rangle$  and  $\langle \alpha^2 p \rangle$  as the average density and pressure in the domain, the effective energy density and pressure can be defined as

$$\frac{8\pi G}{3} \bar{\rho}_{\text{eff}} = \mathcal{F}_{\mathcal{D}} - \frac{1}{6} (\mathcal{R}_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}^T), \quad \frac{8\pi G}{3} \bar{p}_{\text{eff}} = \frac{1}{3} \frac{6\mathcal{F}_{\mathcal{D}} + \mathcal{R}_{\mathcal{D}} - 4\mathcal{P}_{\mathcal{D}} - 4\beta_{\mathcal{D}}^P - 3\mathcal{Q}_{\mathcal{D}}^T}{6}, \quad (29)$$

with the effective equation of state

$$w_{\text{eff}} = \frac{\bar{p}_{\text{eff}}}{\bar{\rho}_{\text{eff}}} = -\frac{1}{3} \frac{\mathcal{R}_{\mathcal{D}} - 4\mathcal{P}_{\mathcal{D}} - 4\beta_{\mathcal{D}}^P - 3\mathcal{Q}_{\mathcal{D}}^T + 6\mathcal{F}_{\mathcal{D}}}{\mathcal{R}_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}}^T - 6\mathcal{F}_{\mathcal{D}}}. \quad (30)$$

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<sup>2</sup> While this work was in final preparation, a broadly equivalent study [27] was completed. Both formalisms consider a generic spacetime, differing chiefly in their definition of the averaged Hubble rate.

We should however, take care with interpretation. Different factors in the lapse can cause complications – the clearest example is demonstrated in the next section when the use of conformal time and cosmological perturbation theory will introduce powers of the background scale factor in the densities and correction terms, but the issue arises more generally. Likewise, we will be averaging across the inhomogeneous surface defined by a perturbed FLRW universe, and comparing the underlying model with the re-averaged model, effectively calculating  $(\dot{a}_{\mathcal{D}}/a_{\mathcal{D}})^2 - (\dot{a}/a)^2$  and  $\ddot{a}_{\mathcal{D}}/a_{\mathcal{D}} - \ddot{a}/a$ , as in Paper I and Paper II. Some subsidiary steps must be taken to separate terms that might appear in the usual Friedmann equations from pure modifications. As an example, in Paper II the modifications for a perturbed FLRW universe with arbitrary spatial curvature were derived, and the curvature correction  $\mathcal{R}_{\mathcal{D}}$  contains a term  $\mathcal{K}/a^2$ , which appears in the background Friedmann equations for this model. As a result, the effective fluid – collecting the modifications to the underlying FLRW model – will be identified later. In general it seems wise to identify the effective fluid in a model-dependent manner.

### III. BACKREACTION IN PERTURBATION THEORY

#### A. Basic Results

The perturbed Robertson-Walker line element with a scale factor  $a = a(\eta)$  can be written as

$$ds^2 = a^2(\eta) \left( -(1 + 2\phi)d\eta^2 + 2B_i d\eta dx^i + (\delta_{ij} + 2C_{ij}) dx^i dx^j \right) \quad (31)$$

where the perturbations  $\phi$ ,  $B_i$  and  $C_{ij}$  include contributions from all perturbative orders, with  $\phi = \sum_n (1/n!) \phi^{(n)}$ , for example. In this work we will only consider scalar perturbations. The spatial perturbation can be further separated to

$$C_{ab} = -\psi \delta_{ab} + E_{ab} \quad (32)$$

where  $E_a^a = 0$ . In general we will only retain quantities to second-order in the perturbations and neglect all higher-orders.<sup>3</sup> To second order, the perturbations are

$$\phi = \phi_{(1)} + \frac{1}{2}\phi_{(2)}, \quad B_i = B_i^{(1)} + \frac{1}{2}B_i^{(2)}, \quad C_{ij} = C_{ij}^{(1)} + \frac{1}{2}C_{ij}^{(2)}, \quad (33)$$

but except in products we never explicitly expand them. Indices on the perturbations are raised and lowered with the Kronecker delta. Derivatives with respect to the conformal time will be denoted with an overdot and  $H = \dot{a}/a$  is the conformal Hubble rate of the underlying model. Directly associating the spatial coordinates in (31) and (4) and noting that  $\beta^i = h^{ij}\beta_j$  gives

$$h_{ij} = a^2 (\delta_{ij} + 2C_{ij}), \quad h^{ij} = a^{-2} \left( \delta^{ij} - 2C^{ij} + 4C^{ik}C_k^j \right), \quad \beta_i = a^2 B_i, \quad \beta^i = B^i - 2C^{ij} B_j. \quad (34)$$

Since  $\alpha^2 - \beta^a \beta_a = a^2(1 + 2\phi)$ , the lapse is then

$$\alpha^2 = a^2 (1 + 2\phi + B^2) \quad (35)$$

where  $B^2 = B^a B_a$ . Paper II contains some discussion on the use of perturbation theory in backreaction, but some further comments are useful here. In essence, FLRW cosmology involves the use of an implicit ‘‘average’’ background upon which perturbations of arbitrary order are defined. When we apply a spatial averaging formalism to a perturbed FLRW cosmology, we are averaging across the perturbed manifold. Here, we are taking an average across the surface defined by perturbations that include second-order terms. This naturally introduces an apparent ambiguity between the Hubble rate  $\dot{a}/a$  appearing in the usual background equations

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} a^2 \bar{\rho} + \frac{1}{3} a^2 \Lambda, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} a^2 (\bar{\rho} + 3\bar{p}) + \frac{1}{3} a^2 \Lambda + \left( \frac{\dot{a}}{a} \right)^2 \quad (36)$$

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<sup>3</sup> This includes terms of the form  $\langle \Phi^2 \rangle^2$ . In principle, these quantities could be large; however, it is clear that, within the confines of perturbation theory, these quantities are approximately of fourth-order in perturbations.

and in the averaged equations (27, 28). However, the Hubble rate  $\dot{a}_{\mathcal{D}}/a_{\mathcal{D}}$  is the Hubble rate defined by the time variation of an expanding volume living on the perturbed 3-surface, entirely distinct from the Hubble rate of the underlying model. In this paper, as in Paper I and Paper II, we will define the effects of ‘‘backreaction’’ to be the difference between these Hubble rates, and between the acceleration rates  $\ddot{a}_{\mathcal{D}}/a_{\mathcal{D}}$  and  $\ddot{a}/a$ .

From the Ricci scalar on the three-surface, the curvature correction is

$$\begin{aligned} \mathcal{R}_{\mathcal{D}} = & \langle (1 + 2\phi) (2\partial^a \partial^b C_{ab} - 2\partial^a \partial_a C) + 4C_{ab} \partial^a \partial^b C + 4C^{ab} \partial^c \partial_c C_{ab} - 8C_{ab} \partial^c \partial^a C_c^b + 3\partial^a C^{bc} \partial_a C_{bc} \rangle \\ & - \langle 2(\partial^a C^{bc})(\partial_b C_{ac}) - (\partial^a C)(2\partial^b C_{ab} - \partial_a C) + 2(\partial_a C^{ab})(2\partial^c C_{bc} - \partial_b C) \rangle. \end{aligned} \quad (37)$$

The extrinsic curvature is

$$\alpha K_b^a = -\frac{\dot{a}}{a} \delta_b^a - \dot{C}_b^a + \frac{1}{2} (\partial^a B_b + \partial_b B^a) - (\partial^a C_b^c + \partial_b C^{ac} - \partial^c C_b^a) B_c + 2C^{ac} \dot{C}_{bc} - C^{ac} (\partial_c B_b + \partial_b B_c) \quad (38)$$

and so the total kinematical backreaction is

$$\begin{aligned} \mathcal{Q}_{\mathcal{D}}^T = & -4\frac{\dot{a}}{a} \langle \partial^a B_a - B^a (2\partial^b C_{ab} - \partial_a C) - 2C^{ab} \partial_a B_b \rangle - \frac{2}{3} \langle \dot{C} \rangle^2 \\ & + \left\langle \dot{C}^2 - \dot{C}_b^a \dot{C}_a^b + (\partial^a B_a)^2 - \frac{1}{2} (\partial^a B_b)(\partial^b B_a) - \frac{1}{2} (\partial^a B_b)(\partial^a B_b) - 2\dot{C} \partial^a B_a + 2\dot{C}_b^a \partial_a B^b \right\rangle. \end{aligned} \quad (39)$$

The dynamical backreactions become

$$\begin{aligned} \mathcal{P}_{\mathcal{D}} = & 3 \left( \frac{\dot{a}}{a} \right)^2 + \left\langle \partial^a \partial_a \phi + \frac{\dot{a}}{a} (3\dot{\phi} + \dot{C}) - \frac{\dot{a}}{a} \partial^a B_a \right\rangle \\ & + \left\langle B_a \partial^b \partial_b B^a + (\partial^a B_b)(\partial_a B^b) - (\partial^a \phi)(\partial_a \phi) - 2C^{ab} \partial_a \partial_b \phi - (\partial^a \phi)(2\partial^b C_{ab} - \partial_a C) + \dot{\phi} (\dot{C} - \partial^a B_a) \right\rangle \\ & - \frac{\dot{a}}{a} \left\langle 3B_a \dot{B}^a + 6\phi \dot{\phi} + 2C^{ab} \dot{C}_{ab} - 2C^{ab} \partial_a B_b - B^a (2\partial^b C_{ab} - \partial_a C) \right\rangle \end{aligned} \quad (40)$$

and

$$\begin{aligned} \beta_{\mathcal{D}}^P = & \left\langle \partial^a \dot{B}_a + 6\frac{\dot{a}}{a} \partial^a B_a \right\rangle - \left\langle 2C^{ab} \partial_a \dot{B}_b + 2\dot{C}^{ab} \partial_a B_b + 12\frac{\dot{a}}{a} C^{ab} \partial_a B_b + (\partial^a B_a)^2 \right\rangle \\ & - \left\langle \dot{B}^a (2\partial^b C_{ab} - \partial_a C) + B^a (2\partial^b \dot{C}_{ab} - \partial_a \dot{C}) + 6\frac{\dot{a}}{a} B^a (2\partial^b C_{ab} - \partial_a C) \right\rangle \\ & + \left\langle B^a (\partial_a \dot{C} - \partial_a \partial^b \dot{B}_b) + 2\dot{C} \partial^a B_a - 3\frac{\dot{a}}{a} B^a \partial_a \phi \right\rangle. \end{aligned} \quad (41)$$

The dynamical backreaction then includes a background term  $3(\dot{a}/a)^2$  which appears in the background Einstein equations (36). We subtract this from  $\mathcal{P}_{\mathcal{D}}$  and include it in the averaged Raychaudhuri equation in its own right: that is, henceforth  $\mathcal{P}_{\mathcal{D}} \rightarrow \mathcal{P}_{\mathcal{D}} - 3(\dot{a}/a)^2$ . Since there is no gauge choice in which  $B^a = C^{ab} = 0$ , from (37, 39, 40, 41) it is already apparent that there exists no gauge in which all the correction terms vanish.

Finally, we have the fluid corrections. For a perfect fluid with  $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$ , the correction term  $\mathcal{F}_{\mathcal{D}}$  is

$$\mathcal{F}_{\mathcal{D}} = \frac{8\pi G}{3} \langle (\rho + p) \left( (\alpha n^\mu u_\mu)^2 - \alpha^2 \right) \rangle = \frac{8\pi G}{3} \langle (\rho + p) \left( (u_0 - \beta^i u_i)^2 - \alpha^2 \right) \rangle. \quad (42)$$

The rest-frame energy density and pressure are expanded to second-order as

$$\rho = \bar{\rho} + \delta\rho = \bar{\rho}(1 + \delta), \quad p = \bar{p} + \delta p. \quad (43)$$

Here  $\bar{p} = w\bar{\rho}$ , and if we define the speed of sound to be  $c_s^2 = \partial p / \partial \rho$  then the second-order expansion of the pressure perturbation is

$$\delta p = \frac{\partial p}{\partial \rho} \delta\rho + \frac{1}{2} \frac{\partial^2 p}{\partial \rho^2} (\delta\rho)^2 = \bar{p} \left( c_s^2 \delta - \frac{(c_s^2)'}{2(1+w)H} \delta^2 \right) \quad (44)$$

where  $\delta$  includes contributions from all perturbative orders and we have used the background matter continuity equation. With the 4-velocity

$$u^\mu = a^{-1} \left( 1 - \phi + \frac{1}{2} (v^2 + 3\phi^2 - 2B_i v^i), v^i \right) \quad (45)$$

where  $v^2 = \delta_{ij}v^iv^j$ ,  $\mathcal{F}_{\mathcal{D}}$  becomes

$$\mathcal{F}_{\mathcal{D}} = \frac{8\pi G}{3}a^2(\bar{\rho} + \bar{p})\langle(v_a + B_a)(v^a + B^a)\rangle. \quad (46)$$

Expanding  $\langle\alpha^2\rho\rangle$  and  $\langle\alpha^2p\rangle$  with (35) and (43), we can finally express the Buchert equations in a perturbed FLRW universe in an unfix gauge as

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 = \frac{8\pi G}{3}a^2\bar{\rho} + \frac{1}{3}a^2\Lambda - \frac{1}{6}(\mathcal{R}_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}^T - 6\mathcal{T}_{\mathcal{D}} - 6\mathcal{L}_{\mathcal{D}}), \quad (47)$$

$$\left(\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right) = -\frac{4\pi G}{3}a^2(\bar{\rho} + 3\bar{p}) + \frac{1}{3}a^2\Lambda + \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{3}\left(\mathcal{P}_{\mathcal{D}} + \beta_{\mathcal{D}}^P + \mathcal{Q}_{\mathcal{D}}^T + 3\mathcal{L}_{\mathcal{D}} - \frac{3}{2}\mathcal{T}_{\mathcal{D}} - 3\mathcal{S}_{\mathcal{D}}\right) \quad (48)$$

where now

$$\mathcal{T}_{\mathcal{D}} = \frac{8\pi G}{3}a^2\bar{p}\langle\delta + 2\phi + 2\phi\delta + B^2 + (1+w)(v_i + B_i)(v^i + B^i)\rangle, \quad (49)$$

$$\mathcal{S}_{\mathcal{D}} = \frac{4\pi G}{3}a^2\bar{p}\left\langle 3c_s^2\delta + 6w\phi + 6c_s^2\phi\delta + 3wB^2 + (1+w)(v_i + B_i)(v^i + B^i) - \frac{(c_s^2)'}{2(1+w)H}\delta^2 \right\rangle \quad (50)$$

and  $\mathcal{L}_{\mathcal{D}}$  is a correction dependent on the cosmological constant,

$$\mathcal{L}_{\mathcal{D}} = \frac{1}{3}a^2\Lambda\langle 2\phi + B^2 \rangle. \quad (51)$$

The effective fluid giving the differences between the base and re-averaged FLRW model then has the effective energy density and pressure

$$\frac{8\pi G}{3}a^2\bar{\rho}_{\text{eff}} = \frac{6\mathcal{T}_{\mathcal{D}} + 6\mathcal{L}_{\mathcal{D}} - \mathcal{R}_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}}^T}{6}, \quad (52)$$

$$\frac{8\pi G}{3}a^2\bar{p}_{\text{eff}} = \frac{\frac{1}{3}\mathcal{R}_{\mathcal{D}} - 4\mathcal{P}_{\mathcal{D}} - 4\beta_{\mathcal{D}}^P - 3\mathcal{Q}_{\mathcal{D}}^T + 12\mathcal{S}_{\mathcal{D}} - 18\mathcal{L}_{\mathcal{D}}}{6}}{\frac{1}{3}\mathcal{R}_{\mathcal{D}} - 4\mathcal{P}_{\mathcal{D}} - 4\beta_{\mathcal{D}}^P - 3\mathcal{Q}_{\mathcal{D}}^T + 12\mathcal{S}_{\mathcal{D}} - 18\mathcal{L}_{\mathcal{D}}}} \quad (53)$$

and an effective equation of state

$$w_{\text{eff}} = -\frac{\frac{1}{3}\mathcal{R}_{\mathcal{D}} - 4\mathcal{P}_{\mathcal{D}} - 4\beta_{\mathcal{D}}^P - 3\mathcal{Q}_{\mathcal{D}}^T + 12\mathcal{S}_{\mathcal{D}} - 18\mathcal{L}_{\mathcal{D}}}{\mathcal{R}_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}^T - 6\mathcal{T}_{\mathcal{D}} - 6\mathcal{L}_{\mathcal{D}}}}{\frac{1}{3}\mathcal{R}_{\mathcal{D}} - 4\mathcal{P}_{\mathcal{D}} - 4\beta_{\mathcal{D}}^P - 3\mathcal{Q}_{\mathcal{D}}^T + 12\mathcal{S}_{\mathcal{D}} - 18\mathcal{L}_{\mathcal{D}}}}. \quad (54)$$

These can be compared with equations (33-35) of Paper II, generalised to a case with a non-vanishing shift vector and the use of a time coordinate other than coordinate time.

## B. Gauge Choices

The equations presented thus far are in an entirely unfix gauge and can be adapted with relative ease to any perturbed FLRW universe. From the form of the correction terms in general, it is immediately apparent that we cannot find a gauge in which each contribution vanishes. The remaining question is whether we can find a gauge in which the corrections take on the simplest possible form. The most obvious choices are those gauges employed in standard cosmological perturbation theory for their simplicity: the uniform curvature, or flat, gauge; the uniform density gauges; the conformal Newtonian gauge; and the conformal synchronous gauge. Uniform curvature gauge would remove the curvature correction, a uniform density gauge should simplify the fluid corrections, the conformal Newtonian gauge should simplify the anisotropies significantly, and a synchronous gauge simplifies the dynamical backreaction. The gauge transformations and governing equations in FLRW universes perturbed up to second order can be found in [29] and we will not repeat them here.

The uniform curvature gauge is of immediate interest in backreaction. In this gauge the spatial surfaces align with those of the background FLRW model and the inhomogeneities are carried entirely by the choice of threading. As a result, by definition the curvature correction  $\mathcal{R}_{\mathcal{D}}$  vanishes in this gauge, although this is balanced against the introduction of the additional dynamical backreaction  $\beta_{\mathcal{D}}^P$  and shift terms in the kinematical backreaction. No other

trivial choice of gauge entirely removes one of the backreaction terms when the system is written in conformal time; even for conformal synchronous gauge there is a non-trivial contribution to  $\mathcal{P}_{\mathcal{D}}$  arising from the time-derivative of  $\alpha$ .

The conformal Newtonian gauge is likewise simple to motivate – while it doesn't remove any of the individual correction terms, the perturbation to the 3-metric is diagonalised and the shift is vanishing, which in particular considerably simplifies  $\mathcal{R}_{\mathcal{D}}$  from the general case. It is also straightforward to find analytical solutions for simple systems in conformal Newtonian gauge, justifying its study (see for example [30, 31]).

Another interesting choice is the uniform density gauge. Here the slicing is defined by choosing  $\delta\rho = 0$ , and the threading by setting for example  $E = 0$  in (32). In this gauge, then, the velocity, shift and lapse are all non-vanishing. We can immediately see that the curvature correction will take the same form as that in conformal Newtonian gauge but that the kinematical and dynamical backreactions are significantly more complicated than those in either of the other two gauges. Furthermore, the aim of a uniform density gauge would be to simplify the fluid corrections  $\mathcal{T}_{\mathcal{D}}$  and  $\mathcal{S}_{\mathcal{D}}$ , and this it fails to do. Setting  $\delta = 0$  leaves the fluid corrections as unwieldy combinations of the lapse, shift and velocity.

The final obvious choice is conformal synchronous gauge. Here we will not consider this gauge, since the curvature correction  $\mathcal{R}_{\mathcal{D}}$  is complicated (in real space) by the anisotropic components of the metric perturbation, and it has been well studied in the literature before (as for example in [13, 16]). However, conformal synchronous gauge does have the advantage that the scale factor  $a_{\mathcal{D}}$  corresponds to a volume containing constant mass and therefore perhaps has a clearer physical meaning than in the other gauges.<sup>4</sup>

We will hence focus on the two choices that seem most likely to yield simple results: the uniform curvature and the conformal Newtonian gauges. In the current work it is not our aim to solve the resulting equations; instead, we are considering the form of the corrections in two particular, simple gauge choices. We will, however, briefly consider pure dust universes in both cases.

### 1. Uniform Curvature Gauge

The uniform curvature gauge is of direct interest since the spatial hypersurfaces align with those of the FLRW background. The volume element on the hypersurfaces in flat gauge, and therefore the spatial averaging across those surfaces, then coincide with the FLRW background.

The metric in the uniform curvature gauge is found by setting  $C_{ab} = 0$  in the above. Then

$$ds^2 = a^2(\eta) \left( -(1 + 2\phi)d\eta^2 + 2B_i d\eta dx^i + \delta_{ij} dx^i dx^j \right) \quad (55)$$

and so the spatial metric, shift and lapse are

$$h_{ij} = a^2 \delta_{ij}, \quad \beta^i = B^i, \quad \alpha^2 = a^2(1 + 2\phi + B^2). \quad (56)$$

This gauge being spatially flat, the curvature correction is trivially

$$\mathcal{R}_{\mathcal{D}} = 0. \quad (57)$$

The kinematical backreaction (39) becomes

$$\mathcal{Q}_{\mathcal{D}}^T = \left\langle (\partial^a B_a)^2 - \frac{1}{2} (\partial^a B_b) (\partial_a B^b) - \frac{1}{2} (\partial_a B^b) (\partial^b B_a) - 4 \frac{\dot{a}}{a} \partial^a B_a \right\rangle. \quad (58)$$

Subtracting off the Hubble term, the dynamical backreaction (40, 41) is

$$\mathcal{P}_{\mathcal{D}} = \left\langle \partial^a \partial_a \phi + B^a \partial^b \partial_b B_a + (\partial^a B_b) (\partial_a B^b) - (\partial_a \phi) (\partial^a \phi) \right\rangle - \left\langle \dot{\phi} \partial^a B_a \right\rangle - \frac{\dot{a}}{a} \left\langle \partial^a B_a - 3\dot{\phi} - 3B_a \dot{B}^a + 6\phi \dot{\phi} \right\rangle, \quad (59)$$

$$\beta_{\mathcal{D}}^P = \left\langle \partial^a \dot{B}_a - (\partial^a B_a)^2 - B^a \partial_a \partial^b B_b \right\rangle + 3 \frac{\dot{a}}{a} \left\langle 2\partial^a B_a - B^a \partial_a \phi \right\rangle. \quad (60)$$

In flat gauge the fluid velocity takes the form

$$u^\mu = \frac{1}{a} \left( 1 - \phi + \frac{1}{2} (v^2 + 2B_i v^i + 3\phi^2), v^i \right) \quad (61)$$

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<sup>4</sup> The authors are grateful to Dominik Schwarz for highlighting this point.



and so the fluid corrections become

$$\mathcal{T}_{\mathcal{D}} = \frac{8\pi G}{3} a^2 \bar{\rho} \langle \delta + 2\phi + 2\phi\delta + B^2 + (1+w)(v_a + B_a)(v^a + B^a) \rangle \quad (62)$$

and

$$\mathcal{S}_{\mathcal{D}} = \frac{4\pi G}{3} a^2 \bar{\rho} \left\langle 3c_s^2 \delta + 6w\phi + 6c_s^2 \phi\delta + 3wB^2 + (1+w)(v_a + B_a)(v^a + B^a) - \frac{(c_s^2)'}{2(1+w)H} \delta^2 \right\rangle. \quad (63)$$

The correction arising from the cosmological constant is

$$\mathcal{L}_{\mathcal{D}} = \frac{1}{3} a^2 \Lambda \langle 2\phi + B^2 \rangle. \quad (64)$$

We can now consider a pure dust (here termed an Einstein-de Sitter) universe, in which  $w = c_s^2 = \Lambda = 0$ . This case has the dual benefits of being one of the simplest cosmologies, and of being a reasonable approximation to the real universe for a range of redshifts from  $z \approx 1000$  to  $z \approx 1$ . In this universe, the fluid corrections simplify slightly to

$$\mathcal{T}_{\mathcal{D}} = \frac{8\pi G}{3} a^2 \bar{\rho} \langle \delta + 2\phi + 2\phi\delta + B^2 + (v_a + B_a)(v^a + B^a) \rangle, \quad \mathcal{S}_{\mathcal{D}} = \frac{4\pi G}{3} a^2 \bar{\rho} \langle (v_a + B_a)(v^a + B^a) \rangle, \quad \mathcal{L}_{\mathcal{D}} = 0. \quad (65)$$

We see that, despite the vanishing of  $\mathcal{R}_{\mathcal{D}}$ , the system in uniform curvature gauge is still rather complex. A reduction to Fourier space simplifies it somewhat. Adopting the Fourier convention

$$A_{\mathbf{k}} = \int A(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}, \quad A(\mathbf{x}) = \int A_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{d^3\mathbf{k}}{(2\pi)^3} \quad (66)$$

and writing the shift and velocity as

$$v_a = i\hat{k}_a v, \quad B_a = i\hat{k}_a B \quad (67)$$

allows us to write the kinematical backreaction as

$$\mathcal{Q}_{\mathcal{D}}^T = -4 \frac{\dot{a}}{a} \left\langle \int k B_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle. \quad (68)$$

In uniform curvature gauge, then, the entire kinematical backreaction is carried on averages of pure perturbations, and assuming these to vanish (as in Papers I and II) would unsurprisingly lead to an incomplete answer. This is the most important result of the calculation in this gauge. The other correction terms become

$$\begin{aligned} \mathcal{P}_{\mathcal{D}} &= \left\langle \int \left( 3 \frac{\dot{a}}{a} \dot{\phi}_{\mathbf{k}} - 4 \frac{\dot{a}}{a} k B_{\mathbf{k}} - k^2 \phi_{\mathbf{k}} \right) e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle \\ &\quad + \left[ \frac{1}{2} \left\langle \int \left( 3 \frac{\dot{a}}{a} B_{\mathbf{k}} \dot{B}_{\mathbf{k}}^* - 6 \frac{\dot{a}}{a} \phi_{\mathbf{k}} \dot{\phi}_{\mathbf{k}}^* - k \dot{\phi}_{\mathbf{k}} B_{\mathbf{k}}^* \right) \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle + \text{c.c.} \right] - \left\langle \int k^2 |\phi_{\mathbf{k}}|^2 \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle, \end{aligned} \quad (69)$$

$$\beta_{\mathcal{D}}^P = \left\langle \int k \left( \dot{B}_{\mathbf{k}} + 6 \frac{\dot{a}}{a} B_{\mathbf{k}} \right) e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle + 3 \frac{\dot{a}}{a} \left[ \left\langle \int k B_{\mathbf{k}} \dot{\phi}_{\mathbf{k}}^* \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle + \text{c.c.} \right], \quad (70)$$

$$\mathcal{T}_{\mathcal{D}} = \frac{\dot{a}}{a} \left\langle \int (\delta_{\mathbf{k}} + 2\phi_{\mathbf{k}}) \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle + \frac{\dot{a}}{a} \left[ \left\langle \int (\phi_{\mathbf{k}} \delta_{\mathbf{k}}^* + v_{\mathbf{k}} B_{\mathbf{k}}^*) \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle + \text{c.c.} \right] + \frac{\dot{a}}{a} \left\langle \int (2|B_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2) \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle, \quad (71)$$

$$\mathcal{S}_{\mathcal{D}} = \frac{1}{2} \frac{\dot{a}}{a} \left[ \left\langle \int v_{\mathbf{k}} B_{\mathbf{k}}^* \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle + \text{c.c.} \right] + \frac{\dot{a}}{a} \left\langle \int (|B_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2) \frac{d^3\mathbf{k}}{(2\pi)^3} \right\rangle. \quad (72)$$

While analytic solutions in this gauge can be presented, they lack the clarity of those performed in conformal Newtonian gauge, to which we now turn.

## 2. Conformal Newtonian Gauge

In Paper I and Paper II, along with for example [21], the Newtonian gauge was employed, as an alternative to the common use in studies of averaging of synchronous gauge. Newtonian gauge has certain advantages: the gravitational

potential and spatial curvatures have clear physical interpretations and, moreover, remain small across all the scales studied. The Newtonian gauge also possesses a vanishing shift and a diagonal spatial perturbation, which simplifies the calculations somewhat. Here we are considering conformal Newtonian gauge, which differs from the Newtonian gauge in its definition of the time coordinate (or, equivalently, the lapse being scaled by  $a^2$ ). The major advantage of conformal Newtonian gauge over Newtonian gauge is the simplicity in finding solutions – in Paper II the analytic solutions employed were derived in conformal Newtonian gauge – but it also makes it easier to transfer between gauges by incorporating it in a unified framework.

Conformal Newtonian gauge is found by substituting  $\phi \rightarrow \Psi$ ,  $B_i \rightarrow 0$  and  $C_{ab} \rightarrow -\Phi\delta_{ab}$  in the general expressions, and the line element is

$$ds^2 = a^2(\eta) \left( -(1 + 2\Psi)d\eta^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j \right). \quad (73)$$

The kinematical backreaction becomes

$$\mathcal{Q}_{\mathcal{D}}^T = 6 \langle \dot{\Phi} \rangle^2. \quad (74)$$

Again subtracting the Hubble term, the dynamical backreaction is

$$\mathcal{P}_{\mathcal{D}} = \left\langle \partial^a \partial_a \Psi + 3 \frac{\dot{a}}{a} (\dot{\Psi} - \dot{\Phi}) \right\rangle + \langle 2\Phi \partial^a \partial_a \Psi - (\partial_a \Psi) (\partial^a \Psi + \partial^a \Phi) \rangle - 3 \left\langle \dot{\Phi} \dot{\Psi} + 2 \frac{\dot{a}}{a} \Psi \dot{\Psi} + 2 \frac{\dot{a}}{a} \Phi \dot{\Phi} \right\rangle \quad (75)$$

and the curvature correction is

$$\mathcal{R}_{\mathcal{D}} = 4 \langle \partial^a \partial_a \Phi \rangle + \langle 6 (\partial^a \Phi) (\partial_a \Phi) + 8 (\Psi + 2\Phi) \partial^a \partial_a \Phi \rangle. \quad (76)$$

The fluid corrections are

$$\mathcal{T}_{\mathcal{D}} = \frac{8\pi G}{3} a^2 \bar{\rho} \langle \delta + 2\Psi + 2\Psi\delta + (1+w)v^2 \rangle \quad (77)$$

and

$$\mathcal{S}_{\mathcal{D}} = \frac{4\pi G}{3} a^2 \bar{\rho} \left\langle 3c_s^2 \delta + 6w\Psi + 6c_s^2 \Psi\delta + (1+w)v^2 - \frac{(c_s^2)'}{2H(1+w)} \delta^2 \right\rangle, \quad (78)$$

and the contribution from the cosmological constant is simply

$$\mathcal{L}_{\mathcal{D}} = \frac{2}{3} a^2 \Lambda \langle \Psi \rangle. \quad (79)$$

These closely resemble those presented in Paper II, with only  $\mathcal{P}_{\mathcal{D}}$  exhibiting minor differences from the change in time coordinate.

In the particular case of a pure Einstein-de Sitter universe, the vanishing anisotropic stress implies that  $\Phi = \Psi$  and that  $\dot{\Phi} = 0$ . The fluid corrections become

$$\mathcal{T}_{\mathcal{D}} = \frac{8\pi G}{3} a^2 \bar{\rho} \langle \delta + 2\Psi + 2\Psi\delta + v^2 \rangle, \quad \mathcal{S}_{\mathcal{D}} = \frac{4\pi G}{3} a^2 \bar{\rho} \langle v^2 \rangle, \quad \mathcal{L}_{\mathcal{D}} = 0, \quad (80)$$

while the dynamical backreaction simplifies slightly to

$$\mathcal{P}_{\mathcal{D}} = \langle \partial^a \partial_a \Psi \rangle + \langle 2\Phi \partial^a \partial_a \Psi - (\partial_a \Psi) (\partial^a \Psi + \partial^a \Phi) \rangle. \quad (81)$$

The solutions to the linear system in Fourier space are given [30] by

$$\Phi_{\mathbf{k}} = \text{const.}, \quad \delta = -\frac{1}{6} \Phi_{\mathbf{k}} k^2 \eta^2, \quad v = \frac{1}{3} \Phi_{\mathbf{k}} k \eta, \quad a = a_0 \left( \frac{\eta}{\eta_0} \right)^2. \quad (82)$$

Noting as a result of (66) that

$$\langle (\partial^a \Phi) (\partial_a \Phi) \rangle = -\langle \Phi \partial^a \partial_a \Phi \rangle, \quad (83)$$

one can find using these solutions that the effective energy density, pressure and equation of state of the backreaction from second-order perturbations are

$$\frac{8\pi G}{3} a^2 \bar{\rho}_{\text{eff}} = \frac{1}{6} \langle L \rangle + \frac{19}{9} \langle Q \rangle, \quad \frac{8\pi G}{3} a^2 \bar{p}_{\text{eff}} = \frac{1}{27} \langle Q \rangle, \quad w_{\text{eff}} = \frac{1}{57 + 9r/2} \quad (84)$$

where

$$r = \frac{\langle L \rangle}{\langle Q \rangle} \quad (85)$$

is the ratio of the impact from terms linear in the perturbations, and

$$L = \frac{24}{\eta^2} \int (\delta_{\mathbf{k}} + 2\Phi_{\mathbf{k}}) e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{d^3\mathbf{k}}{(2\pi)^3} - 4 \int k^2 \Phi_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{d^3\mathbf{k}}{(2\pi)^3}, \quad Q = \int k^2 |\Phi_{\mathbf{k}}|^2 \frac{d^3\mathbf{k}}{(2\pi)^3} \quad (86)$$

are the linear and quadratic contributions respectively. (Recall that  $\delta = \delta_{(1)} + (1/2)\delta_{(2)}$ , so the “linear” contributions include terms that are at second order in perturbation theory.) It is not the aim of this study to evaluate the magnitude of  $r$ ; however, some general statements may be made.

The effective pressure is positive definite. If the effective energy density is also positive, then, we have  $r > -38/3$  and to find a backreaction from perturbations that acts to accelerate the universe we additionally require  $w_{\text{eff}} < -1/3$ , implying  $r < -40/3$ . These two conditions do not intersect and we can conclude that in a pure dust FLRW universe perturbed to second-order in perturbations, a backreaction with positive effective energy density will never act as a dark energy. If the effective energy density were instead negative, then  $r < -38/3$ , and the backreaction would act as a dark energy for  $w_{\text{eff}} > -1/3$ . This holds for  $r > -38/3$  or  $r < -40/3$ , and so if

$$r < -\frac{40}{3} \quad (87)$$

then the backreaction arising in a pure dust universe will act as a dark energy, albeit one with a negative effective energy density. This requires that the impact from averages of pure perturbations be an order of magnitude greater than that from the quadratic modes.

If instead, as seems likely, the average of linear modes vanishes on large scales due to their Gaussian nature, and the average of second-order modes is of the same order of magnitude as that from quadratic terms, we have

$$r \sim \mathcal{O}(1). \quad (88)$$

For  $r \in [-1, 1]$ , we find that the effective equation of state lies between

$$\frac{1}{72} < w_{\text{eff}} < \frac{1}{53} \quad (89)$$

and in the case when the averages of second-order perturbations can themselves be entirely neglected we recover

$$w_{\text{eff}} = \frac{1}{57} \quad (90)$$

as found in Paper II. From this we can conclude that, for any system where  $\mathcal{O}(r) \approx 1$ , the equation of state of backreaction from perturbations, up to and including second-order modes, is similar to warm dust and approximately

$$w_{\text{eff}} \approx \frac{1}{60}. \quad (91)$$

Only for situations where  $r \lesssim -14$  does the effective equation of state become such that the backreaction acts as a dark energy. This could be possible, for example, on smaller scales when the averages over linear modes cannot be neglected.

#### IV. DISCUSSION

In this paper we have formulated the averaged cosmological equations in the most general case for which the 3+1 separation holds and considered the nonlinear regime of perturbation theory in an unfixed and in two simple gauges. The formalism we have derived is entirely general for any spacetime in any coordinate system, and with any combination of fluids. This formalism differs from that recently presented in [27] as we have defined the Hubble rate by the change of a volume defined on an arbitrary surface, while the author of [27] defined his Hubble rate from the expansion scalar. Before specialising to a perturbative system, we also highlighted some issues with the interpretation of the “backreaction”, depending on the choice of underlying system and of the lapse. These issues are illustrated with reference to perturbative models: in such systems the backreaction terms will in general include terms that otherwise

appear in the background Friedmann equations; and for systems in a conformal time the energy densities should all be scaled by the conformal factor. It is therefore sensible to suggest that general statements concerning the “effective fluid” should be made with care, and that this effective fluid is best defined on a model-specific basis.

One of the motivations for the development of this formalism, other than future study into fully inhomogeneous systems, was to consider the gauge issue in backreaction and, in particular, whether the backreaction in a perturbative model can be removed through a judicious choice of gauge, or at least whether a choice can be found that renders the correction terms in their simplest form. To this end, we applied the general formalism to a Robertson-Walker universe perturbed to second-order in metric and fluid perturbations, working in an unspecified gauge. The resulting averaged equations can then be put into a required gauge by removing the redundant degrees of freedom. We then argued that the most obvious gauge choices are those that either remove or maximally simplify one or more of the backreaction quantities, with the simplest choices being the uniform curvature gauge, the conformal synchronous gauge, and the conformal Newtonian gauge. The system in uniform curvature gauge remains somewhat unwieldy despite the vanishing spatial curvature, with the kinematical and dynamical backreactions complicated instead with contributions from the shift vector. The system appears to reduce to a simpler form in the conformal Newtonian gauge, in which the anisotropic terms and the shift vanish and in which analytic solutions for simple cases are easily found. Working in conformal Newtonian gauge, we generalised the analytical estimates of Paper II to the case where contributions of the form  $\langle \Phi \rangle$  cannot be neglected, finding that, on large scales where the average of linear modes vanishes and that of second-order modes is of the same order as quadratic terms, the backreaction has an equation of state similar to warm dust. On smaller scales where the average of linear modes no longer vanishes it remains possible to argue for an equation of state more drastically different from dust, but the linear contributions would need to be more than an order of magnitude greater than those from quadratic terms.

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