

Loop Quantum Cosmology on a Torus

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Abstract. In this paper we study the effect of a torus topology on Loop Quantum Cosmology. We first derive the Teichmüller space parametrizing all possible tori using Thurston's theorem and construct a Hamiltonian describing the dynamics of these torus universes. We then compute the Ashtekar variables for a slightly simplified torus such that the Gauss constraint can be solved easily. We perform a canonical transformation so that the holonomies along the edges of the torus reduce to a product between almost and strictly periodic functions of the new variables. The drawback of this transformation is that the components of the densitized triad become complicated functions of these variables. Nevertheless we find two ways of quantizing these components, which in both cases leads surprisingly to a continuous spectrum.

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1. Introduction

The Einstein field equations are local equations in the sense that they only describe the local geometry of the spacetime. For example the Robertson-Walker metric explicitly contains the parameter k which gives an account of the intrinsic spatial curvature. Using the Friedmann equations this parameter can be determined experimentally since it is directly related to the density parameter Ω_{tot} and the Hubble parameter h . Recent measurements of the energy density of the universe tend to slightly favor a positively curved universe [1], yet a flat curvature lies within the $1\text{-}\sigma$ range. The most direct conclusion is that the spatial topology of the universe is just \mathbb{R}^3 which is the assumption of the Λ CDM model. Nevertheless in the mathematical literature it is well known that a flat space does not mean that its topology is necessarily \mathbb{R}^3 , in fact there are 18 possible flat topologies. Since the Einstein field equations are not sensitive to topology every possibility has to be considered as a possible candidate for the global geometry of our universe until it is ruled out by experiment. In order to do so we first note that the spectrum of the Laplace operator sensitively depends on the topology, i.e. it is discrete if the eigenstates are normalizable and continuous if not. In the first case the solution for e.g. a torus is given by plane waves with a wave vector \vec{k}_n taking only discrete values $n \in \mathbb{N}$ while in the second case the (weak) solution to the eigenvalue equation is given by the (distributional) plane waves with a continuous wave vector \vec{k} . For example, the eigenvalue problem for Δ on \mathbb{T}^3 is given by $(\Delta + E_{\vec{n}})\Psi_{\vec{n}} = 0$, $\vec{n} \in \mathbb{N}^3$, and on S^3 by $\Delta\Psi_{\beta,l,m} = (\beta^2 - 1)\Psi_{\beta,l,m}$, where $\beta \in \mathbb{N}$, $0 \leq l \leq \beta - 1$ and $|m| \leq l$. The implication of a solution of the form $\Psi_{\vec{n}}$ is the existence of a wave function Ψ_n with a maximum length corresponding to e.g. the length of the edges of the torus. Since the departure from a continuous solution is biggest for large wavelengths we have to look for large-scale structures of the universe in order to distinguish between cosmic topologies. The best way to do so is to measure the inhomogeneities of the cosmic microwave background (CMB), expand these in multipole moments and compare the low multipoles with the predictions from theory. It can be shown that in certain closed topologies a suppression in the power spectrum of the low multipoles is expected because of the existence of a largest wavelength. Since such a suppression is present in the CMB several studies compared the theoretical predictions for various topologies with the data. While most analyzed topologies can already be ruled out three of them describe the data even better than the infinite Λ CDM model, namely the torus [2, 3, 4], the dodecahedron[5, 6] and the binary octahedron[7] (see also references therein). While the last two topologies are spherical the torus is the simplest model of a closed flat topology.

However, we know that standard cosmology cannot be the final answer as its predictability breaks down at the big bang. A quantization of the Friedmann equations a la Wheeler-DeWitt does not improve this behavior either. This situation has changed thanks to a new model called loop quantum cosmology (LQC) developed over the last few years which removes the initial singularity. LQC [8, 9, 10, 11, 12, 13, 14, 15, 16] is the approach motivated by loop quantum gravity (LQG) [17, 18, 19] to the quantization

of symmetric cosmological models. The usual procedure is to reduce the classical phase space of the full theory to a phase space with a finite number of degrees of freedom. The quantization of these reduced models uses the tools of the LQG and is therefore called LQC but it does not correspond to the cosmological sector of LQG. The results of LQC not only provide new insights into the quantum structure of spacetime near the Big-Bang singularity but also remove this singularity by extending the time evolution to negative times.

In sum, on the one hand we have hints from observation that our universe may have a closed topology, on the other hand we have a very successful loop quantization of various cosmologies. Thus, starting from these two motivations, we would like to study LQC with a torus topology. But contrary to the works on the CMB we don't want to restrict the analysis to a cubical torus. To do so we construct a torus using Thurston's theorem and find that the most general torus has six degrees of freedom which consist of e.g. three lengths and three angles. We will study its dynamics by numerically solving the Hamiltonian coupled to a scalar field. After rewriting this Hamiltonian in terms of Ashtekar variables we will see that the quantization of such a torus leads to a product between the standard Hilbert spaces of LQC and the Hilbert spaces over the circle. Moreover, we will find two ways to quantize the components of the triad and show that both (generalized) eigenfunctions are not normalizable in this Hilbert space.

As a side remark we would like to point out that the consequences of putting a non-abelian gauge theory into a box with periodic boundary conditions have been studied in e.g. [20]. The motivation behind this idea is an attempt to explain the quark confinement in QCD without explicitly breaking gauge invariance. To simplify the analysis the $su(N)$ -valued gauge field is chosen to be pure gauge, i.e. $A = U^{-1}dU$ with $U \in SU(N)$, such that the holonomy around a closed curve C only depends on the topological property of C . Since general relativity written in terms of Ashtekar variables is also a (constrained) Yang-Mills theory it may be tentatively to use the methods developed for QCD in a box to LQC of a torus universe. However we will derive an Ashtekar connection for the homogeneous torus which is not pure gauge so that the holonomies along C also depend on the length of C . This may not be surprising in view of the fact that the Hilbert space of LQC on \mathbb{R}^3 is spanned by almost periodic functions with an arbitrary length parameter μ .

This paper is organized as follows: in Section 2 we first introduce the classical dynamics of a torus universe and numerically solve the Friedmann equations with a massless scalar field. In Section 3 we introduce the Ashtekar variables for a torus and also explain the complications that arise because of a closed topology. The loop quantization and the construction of a Hilbert space are explained in Section 4 and Section 5 provides a summary and directions for future works. Appendix A gives a short review of the fundamental domain of the 3-torus and Appendix B describes the dynamics of the torus in terms of Iwasawa coordinates.

2. Compact Homogeneous Universes and their Dynamics

The purpose of this section is to study models in which the spatial section has a compact topology. The compactness of a locally homogeneous space brings new degrees of freedom of deformations, known as Teichmüller deformations. This leads to the conclusion that cosmology on a torus is simply cosmology on \mathbb{R}^3 restricted to a cube may be too naive a point of view, especially since the space of solutions of a torus gets nine additional degrees of freedom, as already mentioned in [21]. We will introduce Teichmüller spaces with an emphasis on a Thurston geometry admitting a Bianchi I geometry as its subgeometry [22, 23, 24, 25, 26] and derive the vacuum Friedmann equations using the Hamiltonian formalism.

2.1. Compact Homogeneous Spaces

Let Σ be a three-dimensional, arcwise connected Riemannian manifold.

Definition 1. *A metric on a manifold Σ is locally homogeneous if $\forall p, q \in \Sigma$ there exist neighborhoods U, V of p resp. q and an isometry $(U, p) \rightarrow (V, q)$. The manifold is globally homogeneous if the isometry group acts transitively on the whole manifold Σ .*

Since Σ is arcwise connected we know that there is a unique universal covering manifold $\tilde{\Sigma}$ up to diffeomorphisms with a metric given by the pullback of the metric on Σ by the covering map

$$\pi : \tilde{\Sigma} \rightarrow \Sigma. \quad (1)$$

Singer [27] proved that the metric on $\tilde{\Sigma}$ is then globally homogeneous and $\tilde{\Sigma}$ is given by $\tilde{\Sigma} \cong \tilde{S}/F$, where \tilde{S} is the orientation preserving isometry group of $\tilde{\Sigma}$ and F its isotropy subgroup.

On the other hand, we can also start from a three-dimensional, simply connected Riemannian manifold $\tilde{\Sigma}$ which admits a compact quotient Σ . In order to construct this compact manifold consider the covering group $\Gamma \subset \tilde{S}$ which is isomorphic to the fundamental group $\pi_1(\Sigma)$ of Σ . This implies that

$$\Sigma = \tilde{\Sigma}/\Gamma,$$

which is Hausdorff iff Γ is a discrete subgroup of \tilde{S} and a Riemannian manifold iff Γ acts freely on $\tilde{\Sigma}$.

Definition 2. *A geometry is the pair $(\tilde{\Sigma}, \tilde{S})$ where \tilde{S} a group acting transitively on $\tilde{\Sigma}$ with compact isotropy subgroup. A geometry $(\tilde{\Sigma}, \tilde{S}')$ is a subgeometry of $(\tilde{\Sigma}, \tilde{S})$ if \tilde{S}' is a subgroup of \tilde{S} . A geometry $(\tilde{\Sigma}, \tilde{S})$ is called maximal if it is not a subgeometry of any geometry and minimal if it does not have any subgeometry.*

We will need the following important theorem:

Theorem 1 (Thurston [28]). *Any maximal, simply connected 3-dimensional geometry which admits a compact quotient is equivalent to the geometry $(\tilde{\Sigma}, \tilde{S})$ where $\tilde{\Sigma}$ is one of E^3 (Euclidean), H^3 (hyperbolic), S_p^3 (3-sphere), $S_p^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL}(2, \mathbb{R})$, Nil or Sol.*

If $(\tilde{\Sigma}, \tilde{S}')$ is not a maximal geometry but is simply connected and admits a compact quotient as well we can find a discrete subgroup Γ' of \tilde{S}' acting freely so as to make $\tilde{\Sigma}/\Gamma'$ compact. Define $(\tilde{\Sigma}, \tilde{S})$ as the maximal geometry with $(\tilde{\Sigma}, \tilde{S}')$ as its subgeometry, i.e. $\tilde{S}' \subset \tilde{S}$. By Thurston's Theorem $(\tilde{\Sigma}, \tilde{S})$ is one of the eight Thurston geometries, which implies that $(\tilde{\Sigma}, \tilde{S}')$ is a subgeometry of one of the eight Thurston geometries.

Theorem 2. *Any minimal, simply connected three-dimensional geometry is equivalent to $(\tilde{\Sigma}, \tilde{S})$, where $\tilde{\Sigma} = \mathbb{R}^3$, $\tilde{S} = \text{Bianchi I-VIII}$; $\tilde{\Sigma} = S_p^3$, $\tilde{S} = \text{Bianchi IX}$; or $\tilde{\Sigma} = S_p^2 \times \mathbb{R}$, $\tilde{S} = SO(3) \times \mathbb{R}$, where S_p^3 is the three-sphere and S_p^2 the two-sphere.*

Let $\text{Rep}(\Sigma)$ denote the space of all discrete and faithful representations $\rho : \pi_1(\Sigma) \rightarrow \tilde{S}$ and the diffeomorphism $\phi : \tilde{\Sigma} \rightarrow \tilde{\Sigma}$ a *global conformal isometry* if $\phi_* \tilde{h}_{ab} = \text{const} \cdot \tilde{h}_{ab}$, where \tilde{h}_{ab} is the spatial metric of the universal covering manifold $\tilde{\Sigma}$. This allows us to define a relation $\rho \sim \rho'$ in $\text{Rep}(\Sigma)$ if there exists a conformal isometry ϕ of $\tilde{\Sigma}$ connected to the identity with $\rho'(a) = \phi \circ \rho(a) \circ \phi'$.

Definition 3. *The Teichmüller space is defined as*

$$\text{Teich}(\Sigma) = \text{Rep}(\Sigma) / \sim$$

with elements called Teichmüller deformations, which are smooth and nonisometric deformations of the spatial metric h_{ab} of Σ , leaving the universal cover $(\tilde{\Sigma}, \tilde{h}_{ab})$ globally conformally isometric.

The situation gets more complicated when we try to extend the previous construction to four-dimensional Lorentzian manifolds. The reason is that the action of the covering group Γ needs to preserve both the extrinsic curvature and the spatial metric of $\tilde{\Sigma}$. Thus we cannot construct a homogeneous compact manifold by the action of a discrete subgroup of \tilde{S} on the spatial three-section $\tilde{\Sigma}$. Instead we need the isometry group of the four-dimensional manifold \tilde{M} . Let $M = \mathbb{R} \times \Sigma$ be a compact homogeneous Lorentzian manifold with metric $g_{\mu\nu}$ and $\tilde{M} = \mathbb{R} \times \tilde{\Sigma}$ its covering with metric $\tilde{g}_{\mu\nu}$ ($\mu, \nu = 0, \dots, 4$).

Definition 4. *Let $(\tilde{\Sigma}, \tilde{h}_{ab})$ be a spatial section of $(\tilde{M}, \tilde{g}_{\mu\nu})$. An extendible isometry is defined by the restriction of an isometry of $(\tilde{M}, \tilde{g}_{\mu\nu})$ on $\tilde{\Sigma}$ which preserves $\tilde{\Sigma}$ and forms a subgroup $\text{Esom}(\tilde{\Sigma})$ of \tilde{S} .*

Thus, in order to get a compact homogeneous manifold from \tilde{M} the covering group Γ must be a subgroup of $\text{Esom}(\tilde{\Sigma})$, i.e.

$$\Gamma \subset \text{Esom}(\tilde{\Sigma}).$$

The line element of $M = \mathbb{R} \times \Sigma$ is given by

$$ds^2 = -dt^2 + h_{ab}(t)\sigma^a\sigma^b,$$

where σ^a are the invariant one-forms.

Therefore the Teichmüller parameters enlarge the parameter space by bringing new degrees of freedom from the deformations defined in Definition 3. In fact, the set of all

possible universal covers $(\tilde{M}, \tilde{g}_{ab})$ carries the degrees of freedom of the local geometry and the covering maps Γ the degrees of freedom of the global geometry which are parameterized by the Teichmüller parameters.

2.2. The Torus Universe

In this section we restrict the above analysis to the case of a flat torus and give only the main results. Further details can be found in [22, 23, 24, 25, 26]. Let $\tilde{M} = \mathbb{R} \times \tilde{\Sigma}$ be the universal cover of M and $\tilde{\Sigma}$ the Thurston geometry $(E^3, ISO(3, \mathbb{R}))$. The isometry group $ISO(3)$ is expressed as $g(\mathbf{x}) = \mathbf{R}\mathbf{x} + \mathbf{a}$, where \mathbf{a} is a constant vector and $\mathbf{R} \in SO(3)$ in order that the orientation be preserved. The Killing vectors of E^3 are

$$\begin{aligned} \xi_1 &= \partial_x, & \xi_2 &= \partial_y, & \xi_3 &= \partial_z, \\ \xi_4 &= -z\partial_y + y\partial_z, & \xi_5 &= -x\partial_z + z\partial_x, & \xi_6 &= -y\partial_x + x\partial_y. \end{aligned}$$

The line element of \tilde{M} is thus given by

$$ds^2 = -dt^2 + \tilde{h}_{ab} dx^a dx^b = -dt^2 + a^2(t) {}^0\tilde{h}_{ab} dx^a dx^b,$$

where ${}^0\tilde{h}_{ab}$ is called the fiducial metric in the LQC literature and dx^a are the invariant one-forms of the group $ISO(3, \mathbb{R})$ ‡.

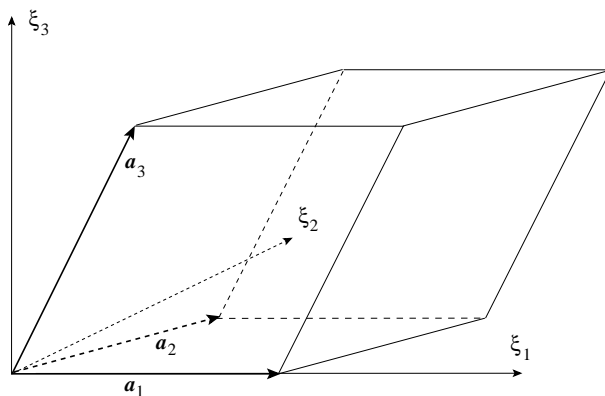


Figure 1. The vectors a_1 , a_2 and a_3 span the torus with six Teichmüller parameters. The global conformal invariance was used in order to align a_1 with ξ_1 and a_2 with $\text{span}\{\xi_1, \xi_2\}$.

The covering group $\Gamma \subset \text{Esom}(\tilde{\Sigma}) \equiv \text{Esom}(E^3)$ allows us to construct a torus via $M = \tilde{M}/\Gamma$, where $M = \mathbb{R} \times \mathbb{T}^3$. The freedom of global conformal transformations allows us to choose the coordinate system of $\tilde{\Sigma}$ such that the generators of the torus have a simple representation. We thus require one of the generators to be aligned with ξ_1 and

‡ When dealing with the open case \mathbb{R}^3 one has to distinguish between the fiducial volume V_0 of a cell as measured by the fiducial metric ${}^0\tilde{h}_{ab}$ and the physical volume V as measured by the physical metric \tilde{h}_{ab} . Since we shall deal with a closed universe we have the "preferred fiducial cell" \mathbb{T}^3 at our disposal. Furthermore, in the open case the spatial integrals have to be restricted to this fiducial cell whereas in the closed case these integrals are naturally restricted to the physical cell \mathbb{T}^3 .

one to lie in the $\xi_1\xi_2$ -plane. The Teichmüller space is then generated by six Teichmüller parameters in three vectors

$$a_1 = \begin{pmatrix} a_1^1 \\ 0 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} a_2^1 \\ a_2^2 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} a_3^1 \\ a_3^2 \\ a_3^3 \end{pmatrix}, \quad (2)$$

where all a_i^j only depend on the coordinate time t . The configuration space \mathcal{C} is therefore spanned by the six Teichmüller parameters such that $\mathcal{C} \subset \mathbb{R}^6$ (see Appendix A). The *flat* spatial metric on \mathbb{T}^3 is then given by ($a, b = 1, 2, 3$)

$$ds^2 = h_{ab}dx^a dx^b, \quad h_{ab} = \sum_c a_a^c a_b^c \quad (3)$$

where

$$(h_{ab}) = \begin{pmatrix} (a_1^1)^2 & a_1^1 a_2^1 & a_1^1 a_3^1 \\ & (a_2^1)^2 + (a_2^2)^2 & a_2^1 a_3^1 + a_2^2 a_3^2 \\ (\text{sym}) & & (a_3^1)^2 + (a_3^2)^2 + (a_3^3)^2 \end{pmatrix}. \quad (4)$$

This metric is invariant under transformations in $SL(3, \mathbb{Z})$. For example it is left invariant by $(a_1^1 \rightarrow -a_1^1, a_2^1 \rightarrow -a_2^1, a_3^1 \rightarrow -a_3^1)$ (see Appendix A for more details). From Equation (4) we can make a Legendre transform of the Einstein-Hilbert action

$$S_{\text{E-H}}[g] = \frac{1}{2\kappa} \int_{\mathbb{R} \times \mathbb{T}^3} *R[g], \quad \kappa = 8\pi G \quad (5)$$

to obtain a Hamiltonian, where $*$ is the Hodge star operator. After a partial integration of \ddot{a}_i^i (which also cancels the surface term we omitted in Equation (5)) we find the following Lagrangian:

$$\begin{aligned} \mathcal{L} = & \frac{1}{4\kappa} \frac{1}{a_1^1 a_2^2 a_3^3} \times \\ & \left[((a_3^2)^2 + (a_3^3)^2) (a_2^1 \dot{a}_1^1 - a_1^1 \dot{a}_2^1)^2 + (a_1^1)^2 (a_3^2)^2 (\dot{a}_2^2)^2 \right. \\ & + (a_2^2)^2 \left\{ (a_3^1)^2 (\dot{a}_1^1)^2 - 2a_1^1 a_3^1 \dot{a}_1^1 \dot{a}_3^1 \right. \\ & \quad \left. \left. + a_1^1 \left(a_1^1 ((\dot{a}_3^1)^2 + (\dot{a}_3^2)^2) - 4a_3^3 \dot{a}_1^1 \dot{a}_3^3 \right) \right\} \right. \\ & - 2a_2^2 \left\{ a_2^1 a_3^2 \dot{a}_1^1 (a_3^1 \dot{a}_1^1 - a_1^1 \dot{a}_3^1) \right. \\ & \quad \left. + a_1^1 \left[a_1^1 a_3^2 \dot{a}_2^1 \dot{a}_3^1 - a_3^1 a_3^2 \dot{a}_1^1 \dot{a}_2^1 \right. \right. \\ & \quad \left. \left. + \dot{a}_2^2 (a_1^1 a_3^2 \dot{a}_3^2 + 2a_3^3 (a_3^3 \dot{a}_1^1 + a_1^1 \dot{a}_3^3)) \right] \right\} \left. \right] \end{aligned}$$

We introduce the momenta

$$p^a{}_b := \frac{\partial \mathcal{L}}{\partial \dot{a}_a^b} \quad (6)$$

conjugate to the configuration variables a_a^b such that the phase space $\mathcal{P} = T^*\mathcal{C} \subset \mathbb{R}^{12}$ is the cotangent bundle over \mathcal{C} with

$$\{a_a^b, p^c{}_d\} = \delta_a^c \delta_d^b, \quad \{a_a^b, a_c^d\} = 0, \quad \{p^a{}_b, p^c{}_d\} = 0, \quad (7)$$

where the Poisson brackets are defined as

$$\{f, g\} = \sum_{a,b=1}^3 \frac{\partial f}{\partial a_a{}^b} \frac{\partial g}{\partial p^a{}_b} - \frac{\partial g}{\partial a_a{}^b} \frac{\partial f}{\partial p^a{}_b}$$

for any smooth functions on the phase space. We insert $\dot{a}_a{}^b = \dot{a}_a{}^b(p^c{}_d)$ into the Legendre transform of Equation (5) and get the Hamiltonian

$$\begin{aligned} \mathcal{H}_g = & \frac{\kappa}{4} \frac{1}{a_1{}^1 a_2{}^2 a_3{}^3} \times \\ & \left[(a_1{}^1 p^1{}_1)^2 + (a_2{}^2 p^2{}_2)^2 + (a_3{}^3 p^3{}_3)^2 + (a_2{}^1 p^2{}_1)^2 + 4(a_2{}^2 p^2{}_1)^2 \right. \\ & + (a_3{}^1 p^3{}_1)^2 + 4(a_3{}^2 p^3{}_1)^2 + 4(a_3{}^3 p^3{}_1)^2 + (a_3{}^2 p^3{}_2)^2 \\ & + 4(a_3{}^3 p^3{}_2)^2 - 2a_3{}^2 a_3{}^3 p^3{}_2 p^3{}_3 \\ & + 2a_1{}^1 p^1{}_1 (a_2{}^1 p^2{}_1 - a_2{}^2 p^2{}_2 + a_3{}^1 p^3{}_1 - a_3{}^2 p^3{}_2 - a_3{}^3 p^3{}_3) \\ & - 2a_3{}^1 p^3{}_1 (a_3{}^2 p^3{}_2 + a_3{}^3 p^3{}_3) \\ & - 2a_2{}^1 p^2{}_1 (a_2{}^2 p^2{}_2 - a_3{}^1 p^3{}_1 + a_3{}^2 p^3{}_2 + a_3{}^3 p^3{}_3) \\ & \left. + 2a_2{}^2 \left\{ a_3{}^2 (4p^2{}_1 p^3{}_1 + p^2{}_2 p^3{}_2) - p^2{}_2 (a_3{}^1 p^3{}_1 + a_3{}^3 p^3{}_3) \right\} \right] \end{aligned} \quad (8)$$

The Hamiltonian constraint $\mathcal{H}_g \approx 0$ reduces the dynamical degrees of freedom from $\dim \mathcal{P} = 12$ to $\dim \mathcal{P} = 10$, which agrees with [21]. To compare this Hamiltonian with the usual Bianchi type I models we set all offdiagonal elements to zero and $a_i{}^i = a_i$, $p^i{}_i = p^i$ (no summation), and get

$$\mathcal{H}_g = \frac{\kappa}{4} \left(\frac{a_1(p^1)^2}{a_2 a_3} + \frac{a_2(p^2)^2}{a_1 a_3} + \frac{a_3(p^3)^2}{a_1 a_2} - 2 \frac{p^1 p^2}{a_3} - 2 \frac{p^2 p^3}{a_1} - 2 \frac{p^1 p^3}{a_2} \right), \quad (9)$$

which agrees with the result given in [31] up to a factor 2 in the definition of the action. To get the isotropic case§ we further set $a_i = a$, $p^i = p/3$ and find that the Hamiltonian (8) reduces to the usual first Friedmann equation

$$\mathcal{H}_g = -\frac{\kappa p^2}{12a}$$

and the Hamiltonian equation $\dot{p}^i{}_j = -\partial \mathcal{H}_g / \partial a_i{}^j$ to the usual second Friedmann equation

$$\dot{p} = -\frac{\partial \mathcal{H}_g}{\partial a} = \frac{\kappa p^2}{12a^2}.$$

The second Hamiltonian equation is given by $\dot{a} = \partial \mathcal{H}_g / \partial p = -\kappa p / (6a)$ and allows us to recast the first Friedmann equation into the usual form

$$\mathcal{H}_g = -3a\dot{a}^2 / \kappa.$$

Furthermore, notice that all $a_i{}^j$ and $p^i{}_j$, $i \neq j$, have to vanish in order for the torus to remain aligned with the Killing fields ξ_I .

§ At this point care has to be taken because there is no homogeneous and isotropic vacuum solution to the Einstein equation (see Section 3.4.1). Only a nonvanishing energy-momentum tensor allows for the isotropic limit of \mathcal{H}_g , which corresponds then to the usual Friedmann solutions.

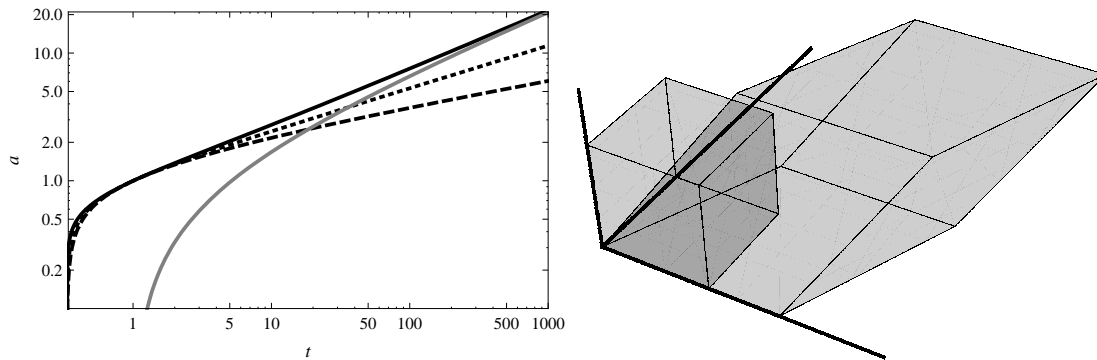


Figure 2. *Left panel:* Solutions corresponding to the Hamiltonian (10) with the initial conditions $a_i^i(1) = 1$, $p^i_i(1) = -1$ (no summation), $a_i^j(1) = 0$ ($i \neq j$), $p^3_1(1) = p^3_2(1) = 0$, $p^2_1(1) = 0.2$, $\phi(1) = 10^{-3}$, $\pi(1) = 1.2$. The diagonal momenta p^i_i are chosen to be negative such that all sides of the torus expand. The solid black line is a_1^1 , the dashed one a_2^2 , the dotted one a_3^3 and the gray one the off-diagonal a_2^1 . The time t parametrizes the coordinate time in natural units ($c = \kappa = \hbar = 1$). *Right panel:* Solution corresponding to the Hamiltonian (10) at two different times. The initial condition is a cubic universe with $a_i^i \equiv a_0$, $p^i_i \equiv p_0$, $a_i^j = 0$ ($i \neq j$), $p^i_j \neq 0$ ($i \neq j$). For both panels the mass and the potential of the scalar field have been set to zero.

We add a matter term consisting of a homogeneous massive scalar field^{||} to Equation (8) to obtain the Hamiltonian

$$\mathcal{H} = \mathcal{H}_g + \mathcal{H}_\phi = \mathcal{H}_g + \frac{1}{2\sqrt{h}}\pi^2 + \frac{\sqrt{h}}{2}m_\phi^2\phi^2 + \sqrt{h}V(\phi), \quad (10)$$

where π is the momentum of the scalar field, m_ϕ its mass, $h = (a_1^1)^2(a_2^2)^2(a_3^3)^2$ the determinant of the spatial metric (4) and $V(\phi)$ the potential which we set to zero in the sequel. From this equation we calculate the Friedmann equations and compute the shape of the universe for a special choice of initial conditions, which is shown in Figure 2. All classical solutions have the limit $\lim_{t \rightarrow 0} a_i^j = 0$ and grow with $a \propto t^{1/3}$ for a massless scalar field with zero potential. Furthermore, note the convergence of a_1^1 and a_2^2 , which is explained in Appendix B.

3. Symmetry Reduction and Classical Phase Space for Ashtekar Variables

In this section we shall repeat the complete analysis introduced in [29, 9, 10] in order to see the role of a compact topology on a connection. Our strategy is to find an invariant connection on the covering space \tilde{M} and then restrict it to the compact space M by

^{||} Notice that since every scalar field lives in the trivial representation of the rotation group it is not possible to construct a scalar field which is homogeneous but anisotropic.

means of the covering map (1). In the following section, when referring to the covering space, we shall use a tilde.

3.1. Invariant Connections

Let $\tilde{P}(\tilde{M}, SU(2), \pi)$ be a principal fiber bundle over \tilde{M} with structure group $SU(2)$ and projection $\pi : \tilde{P} \rightarrow \tilde{M}$. We require that there be a symmetry group $\tilde{S} \subset \text{Aut}(\tilde{P})$ of bundle automorphisms which acts transitively. Furthermore, for Bianchi I models \tilde{S} does not have a non-trivial isotropy subgroup \tilde{F} so that the base manifold is isomorphic to the symmetry group \tilde{S} , i.e. $\tilde{M}/\tilde{S} = \{x_0\}$ is represented by a single point that can be chosen arbitrarily in \tilde{M} . Since the isotropy group \tilde{F} is trivial the coset space $\tilde{S}/\tilde{F} \cong \tilde{S}$ is reductive with a decomposition of the Lie algebra of \tilde{S} according to $\mathcal{L}\tilde{S} = \mathcal{L}\tilde{F} \oplus \mathcal{L}\tilde{F}_\perp = \mathcal{L}\tilde{F}_\perp$ together with the trivial condition $\text{Ad}_{\tilde{F}}\mathcal{L}\tilde{F}_\perp \subset \mathcal{L}\tilde{F}_\perp$. This allows us to use the general framework described in [9, 10, 29].

Since the isotropy group plays an important role in classifying symmetric bundles and invariant connections we describe the general case of a general isotropy group \tilde{F} . Fixing a point $x \in \tilde{M}$, the action of \tilde{F} yields a map $\tilde{F} : \pi^{-1}(x) \rightarrow \pi^{-1}(x)$ of the fiber over x . To each point $p \in \pi^{-1}(x)$ in the fiber we assign a group homomorphism $\lambda_p : \tilde{F} \rightarrow G$ defined by $f(p) =: p \cdot \lambda_p(f)$, $\forall f \in \tilde{F}$. As this homomorphism transforms by conjugation $\lambda_{p \cdot g} = \text{Ad}_{g^{-1}} \circ \lambda_p$ only the conjugacy class $[\lambda]$ of a given homomorphism matters. In fact, it can be shown [29] that an \tilde{S} -symmetric principal bundle $P(\tilde{M}, G, \pi)$ with isotropy subgroup $\tilde{F} \subseteq \tilde{S}$ is uniquely characterized by a conjugacy class $[\lambda]$ of homomorphisms $\lambda : \tilde{F} \rightarrow G$ together with a reduced bundle $Q(\tilde{M}/\tilde{S}, Z_G(\lambda(\tilde{F})), \pi_Q)$, where $Z_G(\lambda(\tilde{F}))$ is the centralizer of $\lambda(\tilde{F})$ in G . In our case, since $\tilde{F} = \{1\}$ all homomorphisms $\lambda : \tilde{F} \rightarrow G = SU(2)$ are given by $1 \mapsto 1_G$.

After having classified the \tilde{S} -symmetric fiber bundle \tilde{P} we seek a $[\lambda]$ -invariant connection on \tilde{P} . We use the following general result [30]:

Theorem 3 (Generalized Wang theorem). *Let \tilde{P} be an \tilde{S} -symmetric principal bundle classified by $([\lambda], Q)$ and let $\tilde{\omega}$ be a connection in \tilde{P} which is invariant under the action of \tilde{S} . Then $\tilde{\omega}$ is classified by a connection $\tilde{\omega}_Q$ in Q and a scalar field (usually called the Higgs field) $\phi : Q \times \mathcal{L}\tilde{F}_\perp \rightarrow \mathcal{L}G$ obeying the condition*

$$\phi(\text{Ad}_f(X)) = \text{Ad}_{\lambda(f)}\phi(X), \forall f \in \tilde{F}, X \in \mathcal{L}\tilde{F}_\perp. \quad (11)$$

The connection $\tilde{\omega}$ can be reconstructed from its classifying structure as follows. According to the decomposition $\tilde{M} \cong \tilde{M}/\tilde{S} \times \tilde{S}/\tilde{F}$ we have $\tilde{\omega} = \tilde{\omega}_Q + \tilde{\omega}_{\tilde{S}/\tilde{F}}$ with $\tilde{\omega}_{\tilde{S}/\tilde{F}} = \phi \circ \iota^* \tilde{\theta}_{\text{MC}}$, where $\iota : \tilde{S}/\tilde{F} \hookrightarrow \tilde{S}$ is a local embedding and $\tilde{\theta}_{\text{MC}}$ is the Maurer-Cartan form on \tilde{S} . The structure group G acts on ϕ by conjugation, whereas the solution space of Equation (11) is only invariant with respect to the reduced structure group $Z_G(\lambda(\tilde{F}))$. This fact leads to a partial gauge fixing since the connection form $\tilde{\omega}_{\tilde{S}/\tilde{F}}$ is a $Z_G(\lambda(\tilde{F}))$ -connection which explicitly depends on λ . We then break down the structure group from G to $Z_G(\lambda(\tilde{F}))$ by fixing a $\lambda \in [\lambda]$.

In our case, the embedding $\iota : \tilde{S} \rightarrow \tilde{S}$ is the identity and the base manifold $\tilde{M}/\tilde{S} = \{x_0\}$ of the orbit bundle is represented by a single point so that the invariant connection is given by

$$\tilde{A} = \phi \circ \tilde{\theta}_{\text{MC}}.$$

The three generators of $\mathcal{L}\tilde{S}$ are given by T_I , $1 \leq I \leq 3$, with the relation $[T_I, T_J] = 0$ for Bianchi I models. The Maurer-Cartan form is given by $\tilde{\theta}_{\text{MC}} = \tilde{\omega}^I T_I$ where $\tilde{\omega}^I$ are the left invariant one-forms on \tilde{S} . The condition (11) is empty so that the Higgs field is given by $\phi : \mathcal{L}\tilde{S} \rightarrow \mathcal{L}G$, $T_I \mapsto \phi(T_I) =: \phi_I^i \tau_i$, where the matrices $\tau_j = -i\sigma_j/2$, $1 \leq j \leq 3$, generate $\mathcal{L}G$, where σ_j are the standard Pauli matrices[¶]. In summary the invariant connection is given by

$$\tilde{A} = \phi_I^i \tau_i d\tilde{\omega}^I. \quad (12)$$

In order to restrict this invariant connection we define the invariant connection A on \mathbb{T}^3 with the pullback given by the covering map (1). The generators of the Teichmüller space (see Equation (2)) allow us to write A as:

$$A_a^i := \bar{\phi}_I^i \omega_a^I, \quad (\bar{\phi}_I^i) = \begin{pmatrix} \bar{\phi}_1^1 & \bar{\phi}_2^1 & \bar{\phi}_3^1 \\ 0 & \bar{\phi}_2^2 & \bar{\phi}_3^2 \\ 0 & 0 & \bar{\phi}_3^3 \end{pmatrix}. \quad (13)$$

3.1.1. Simplified Model In the sequel we shall concentrate on a simpler model for which we can also easily satisfy the Gauss constraint. We choose a torus generated by the vectors $a_1 = (a_1^1, 0, 0)^T$, $a_2 = (0, a_2^2, a_2^3)^T$ and $a_3 = (0, a_3^2, a_3^3)^T$ (see Figure 3) such that

$$(\bar{\phi}_I^i) = \begin{pmatrix} \bar{\phi}_1^1 & 0 & 0 \\ 0 & \bar{\phi}_2^2 & \bar{\phi}_3^2 \\ 0 & \bar{\phi}_2^3 & \bar{\phi}_3^3 \end{pmatrix}, \quad (\omega_a^I) = \begin{pmatrix} a_1^1 & 0 & 0 \\ 0 & a_2^2 & a_3^2 \\ 0 & a_2^3 & a_3^3 \end{pmatrix} \quad (14)$$

and

$$(A_a^i) = \begin{pmatrix} a_1^1 \bar{\phi}_1^1 & 0 & 0 \\ 0 & a_2^2 \bar{\phi}_2^2 + a_2^3 \bar{\phi}_3^2 & a_3^2 \bar{\phi}_2^2 + a_3^3 \bar{\phi}_3^2 \\ 0 & a_2^2 \bar{\phi}_2^3 + a_2^3 \bar{\phi}_3^3 & a_3^2 \bar{\phi}_2^3 + a_3^3 \bar{\phi}_3^3 \end{pmatrix}$$

The vectors X_I dual to ω^I are given by

$$X_1 = \begin{pmatrix} \frac{1}{a_1^1} \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \frac{1}{\mathfrak{h}} \begin{pmatrix} 0 \\ a_3^3 \\ -a_2^3 \end{pmatrix}, \quad X_3 = \frac{1}{\mathfrak{h}} \begin{pmatrix} 0 \\ -a_3^2 \\ a_2^2 \end{pmatrix},$$

where we defined $\mathfrak{h} = a_2^2 a_3^3 - a_2^3 a_3^2$.

[¶] We use the convention $\tau_i \tau_j = \frac{1}{2} \epsilon_{ij}^k \tau_k - \frac{1}{4} \delta_{ij} \mathbb{1}_{2 \times 2}$

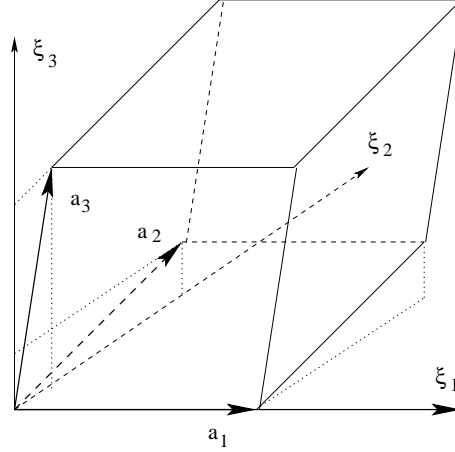


Figure 3. The vectors a_1 , a_2 and a_3 span the torus with five Teichmüller parameters. The vectors a_2 and a_3 lie in the $\xi_2\xi_3$ -plane while a_1 is aligned with ξ_1

3.2. Classical Phase Space for Ashtekar Variables

The phase space of full general relativity in the Ashtekar representation is spanned by the $SU(2)$ -connection $A_a^i = \Gamma_a^i + \gamma K_a^i$ and the densitized triad $E_i^a = |\det e| e_i^a$, where Γ_a^i is the spin connection, K_a^i the extrinsic curvature, e_i^a the triad and $\gamma > 0$ the Immirzi parameter [17, 18, 19]. The symplectic structure of full general relativity is given by the Poisson bracket

$$\{A_a^i(y), E_j^b(x)\} = \kappa \delta_a^b \delta_j^i \delta(x, y). \quad (15)$$

The connection between the metric and the densitized triad is given by

$$hh^{ab} = \delta^{ij} E_i^a E_j^b, \quad (16)$$

where h^{ab} is the inverse of the metric h_{ab} .

We can now use the results obtained in Section 2 to construct the phase space \mathcal{P} in this representation. In the preceding subsection we have already found the configuration space is spanned by $\bar{\phi}_I^i$ (see Equation (14)). On the other hand, the densitized triad dual to the connection is given by

$$(E_i^a) = \sqrt{h} \bar{p}^I{}_i X_I^a = \sqrt{h} \begin{pmatrix} \frac{\bar{p}^1{}_1}{a_1^{\frac{1}{4}}} & 0 & 0 \\ 0 & \frac{a_3^3 \bar{p}^2{}_2 - a_3^2 \bar{p}^3{}_2}{\hbar} & \frac{a_3^3 \bar{p}^2{}_3 - a_3^2 \bar{p}^3{}_3}{\hbar} \\ 0 & \frac{a_2^2 \bar{p}^3{}_2 - a_2^3 \bar{p}^2{}_2}{\hbar} & \frac{a_2^2 \bar{p}^3{}_3 - a_2^3 \bar{p}^2{}_3}{\hbar} \end{pmatrix}, \quad (17)$$

where

$$(\bar{p}^I{}_i) = \begin{pmatrix} \bar{p}^1{}_1 & 0 & 0 \\ 0 & \bar{p}^2{}_2 & \bar{p}^2{}_3 \\ 0 & \bar{p}^3{}_2 & \bar{p}^3{}_3 \end{pmatrix},$$

together with $\omega_a^J X_I^a = \delta_I^J$ and $h = (a_1^1)^2 (a_2^3 a_3^2 - a_2^2 a_3^3)^2$ is the determinant of the spatial metric constructed from the vectors a_i and $\bar{p}^I{}_i$ the momentum dual to $\bar{\phi}_I^i$

satisfying the Poisson bracket

$$\{\bar{\phi}_I^i, \bar{p}^J_j\} = \frac{\kappa\gamma}{V_0} \delta_I^J \delta_j^i \quad (18)$$

with the volume $V_0 = \int_{\mathbb{T}^3} d^3x \sqrt{h}$ of \mathbb{T}^3 as measured by the metric h . For later purpose we define new variables

$$\phi_I^i = L_I \bar{\phi}_I^i, \quad p^I_i = \frac{V_0}{L_I} \bar{p}^I_i, \quad (19)$$

such that

$$\{\phi_I^i, p^J_j\} = \kappa\gamma \delta_I^J \delta_j^i, \quad (20)$$

where

$$L_1 = a_1^{-1}, \quad L_2 = \sqrt{(a_2^{-2})^2 + (a_2^{-3})^2}, \quad L_3 = \sqrt{(a_3^{-2})^2 + (a_3^{-3})^2}.$$

Thus we conclude that

Proposition 1. *The classical configuration space $\mathcal{A}_S = \mathbb{R}^5$ is spanned by the five configuration variables ϕ_I^i . The phase space $\mathcal{P} = \mathbb{R}^{10}$ is spanned by ϕ_I^i and the five momenta p^J_j satisfying the Poisson bracket (20).*

Furthermore, note that the determinant of the densitized triad is given by

$$\det E_i^a = \mathfrak{k} p^1_1 (p^2_3 p^3_2 - p^2_2 p^3_3), \quad (21)$$

where we defined

$$\mathfrak{k} := \frac{L_1 L_2 L_3}{V_0}.$$

The relation between the new variables (ϕ_I^i, p^J_j) and the 'scale factors' a_a^b and their respective momenta p^a_b can be found by using Equation (16) and the Poisson brackets (7) and (20). A closed form could only be found for p^1_1 and is given by

$$|p^1_1| = |a_2^{-2} a_3^{-3} - a_2^{-3} a_3^{-2}|.$$

3.3. Constraints in Ashtekar Variables on the Torus

In the canonical variables (15) the Legendre transform of the Einstein-Hilbert action (5) results in a fully constrained system [17, 18, 19]

$$S = \frac{1}{2\kappa} \int_{\mathbb{R}} dt \int_{\mathbb{T}^3} d^3x \left(2\dot{A}_a^i E_i^a - [\Lambda^j G_j + N^a H_a + N\mathcal{H}] \right), \quad (22)$$

where G_j is the Gauss constraint, H_a the diffeomorphism (or vector) constraint, \mathcal{H} the Hamiltonian and Λ^j , N^a , N are Lagrange multipliers. The Hamiltonian constraint simplifies to

$$C_{\text{grav}} = -\frac{1}{2\kappa} \int_{\mathbb{T}^3} d^3x N \epsilon_{ijk} F_{ab}^i \frac{E^{aj} E^{bk}}{\sqrt{|\det E|}} \quad (23)$$

due to spatial flatness, where the curvature of the Ashtekar connection is given by

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon_{jk}^i A_a^j A_b^k = \epsilon_{jk}^i A_a^j A_b^k.$$

Homogeneity further requires that $N \neq N(x)$. Inserting Equation (14) and Equation (17) into Equation (23) we get

$$C_{\text{grav}} = -\frac{1}{\kappa\gamma^2} \frac{1}{\sqrt{|p^1_1(p^2_2 p^3_3 - p^2_3 p^3_2)|}} \times \left[\phi_1^1 p^1_1 \{(\phi_2^2 - \phi_2^3)(p^2_2 - p^2_3) + (\phi_3^2 - \phi_3^3)(p^3_2 - p^3_3)\} + (\phi_2^3 \phi_3^2 - \phi_2^2 \phi_3^3)(p^2_3 p^3_2 - p^2_2 p^3_3) \right], \quad (24)$$

where we defined $N = \sqrt{L_1 L_2 L_3 / V_0}$ in order to simplify the Hamiltonian. Using the Hamiltonian (24) we can compute the time evolution of the basic variables ϕ_i^j and p^i_j (see Figure 4). Setting all off-diagonal terms to zero we see that Equation (24) matches with Eq. (3.20) in [31]. If we further set $\phi_{(i)}^i = c$ and $p^{(i)}_i = p$ we get

$$C_{\text{grav}} = -\frac{3}{\kappa\gamma^2} c^2 \sqrt{|p|},$$

which is exactly the same result as the homogeneous and isotropic case [8].

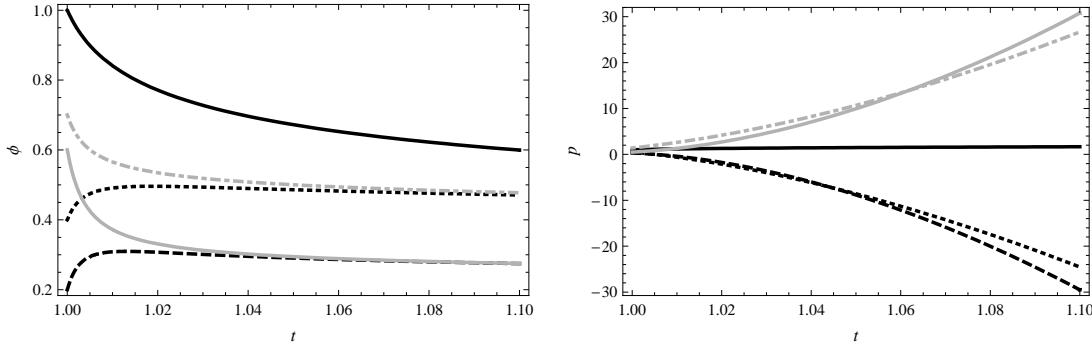


Figure 4. Solutions corresponding to the Hamiltonian (24) coupled to a massless scalar field with vanishing potential. *Left panel:* the black thick solid shows the evolution of ϕ_1^1 , ϕ_2^2 is the black dashed line, ϕ_3^3 the dotted line, ϕ_2^3 the gray dashdotted one and ϕ_3^2 the solid gray one. *Right panel:* the black thick solid shows the evolution of p^1_1 , p^2_2 is the black dashed line, p^3_3 the dotted line, p^2_3 the gray dashdotted one and p^3_2 the solid gray one. In both cases the initial conditions are $\phi_1^1 = 1.0$, $\phi_2^2 = 0.2$, $\phi_3^3 = 0.4$, $\phi_2^3 = 0.6$, $\phi_3^2 = 0.7$, $p^1_1 = 1.0$, $p^2_2 = 0.3$, $p^3_3 = 0.5$, $p^2_3 = 0.5$, $p^3_2 = 1.4$, $\phi = 0.01$ and $p_\phi = 8.1$. The time t parametrizes the coordinate time in natural units ($c = \kappa = \hbar = 1$).

3.4. Diffeomorphism and Gauss Constraints

The Gauss constraint stems from the fact that we chose the densitized triads E_i^a as the momenta conjugated to the connections A_a^i . In fact, the spacial metric can be directly obtained from the densitized triads through Equation (16) and is invariant under rotations given by $E_i^a \mapsto O_i^j E_j^a$. In order that the theory be invariant under such rotations the Gauss constraint

$$G_i = \partial_a E_i^a + \epsilon_{ijk} A_a^j E_k^a \approx 0 \quad (25)$$

must be satisfied. The diffeomorphism constraint modulo Gauss constraint originates from the requirement of independence from any spatial coordinate system or background and is given by

$$H_a = F_{ab}^i E_i^b \approx 0. \quad (26)$$

However, as mentioned in [21] we have to be careful when dealing with these constraints in the case where the topology is closed. We thus divide this subsection into two parts, starting with the general case of open models.

3.4.1. Open Models Due to spatial homogeneity of Bianchi type I models the basic variables can be diagonalized to [9, 14]

$$A_a^i = \tilde{c}'_{(K)} \Lambda_K^i \omega_a^K, \quad E_i^a = p'^{(K)} \Lambda_i^K X_K^a,$$

where ω' is the left-invariant 1-form, X' the densitized left-invariant vector field dual to ω' and $\Lambda' \in SO(3)^+$. This choice of variables automatically satisfies the vector and Gauss constraints, thus reducing the analysis of Equation (22) to the Hamiltonian constraint (23). The homogeneous, anisotropic vacuum solution to the Einstein field equations is called the Kasner solution and is given by the following metric:

$$ds^2 = -d\tau^2 + \tau^{2\alpha_1} dx_1^2 + \tau^{2\alpha_2} dx_2^2 + \tau^{2\alpha_3} dx_3^2$$

where the two constraints $\alpha_i \in \mathbb{R}$, $\sum \alpha_i = \sum \alpha_i^2 = 1$ have to be fulfilled. These imply that not all Kasner exponents can be equal, i.e. isotropic expansion or contraction of space is not allowed. By contrast the RW metric is able to expand or contract isotropically because of the presence of matter. At the end, from the twelve-dimensional phase space only two degrees of freedom remains.

An infinitesimal diffeomorphism generated by a vector field V induces the following action on the left-invariant 1-form ω' :

$$\omega_a \mapsto \omega'_a + \epsilon \mathcal{L}_V \omega'_a, \quad (27)$$

where \mathcal{L}_V is the Lie derivative along V . Such transformations leave the metric homogeneous provided the vector fields satisfy

$$V^a = -(f_j^i y^j) X_i^a \quad (28)$$

for some constants f_j^i and functions y^i given by $\mathcal{L}_{K_j} y^i = \delta_j^i$ [21]. The last equation for y^i relies on the fact that the 3-surface is topologically \mathbb{R}^3 and the Killing vectors K_i commute. As we shall see below this will not be the case in the closed models.

In the case of rotational symmetry the diffeomorphism constraint is once again satisfied by the choice of variables whereas the Gauss constraint is not. However, in such a case the triad components can be rotated until the Gauss constraint is also satisfied. Further details can be found in [32].

⁺ In order to avoid confusion with the rest of this work we tag every variable with a "''" when dealing with the open case.

3.4.2. The Torus as a Closed Model As we have seen in Section 2 it is not possible to align the Killing fields with the left-invariant vectors, whence the metric takes the non-diagonal form (4) and the Ashtekar connection the form (13). In the previous subsection we saw that a diffeomorphism preserves homogeneity provided it satisfies the condition (28). In the closed model the analysis goes through as well and we find that V_i has to satisfy the same condition (28). However, since such fields lack the required periodicity in x^i we are led to the conclusion that there are no globally defined, non-trivial homogeneity preserving diffeomorphisms (HPDs) and there is no analog of (27). Thus, instead of one degree of freedom we get additional degrees of freedom.

The Gauss constraint for a Bianchi type I model is given by

$$G_i = \epsilon_{ijk} \phi_I^i p^I_k. \quad (29)$$

With our choice of variables two Gauss constraints are automatically satisfied, namely $G_2 = G_3 \equiv 0$. However, we can still perform a global $SU(2)$ transformation along τ_1 which is implemented in the nonvanishing Gauss constraint

$$G_1 = \phi_2^2 p^2_3 + \phi_3^2 p^3_3 - \phi_2^3 p^2_2 - \phi_3^3 p^3_2 \approx 0 \quad (30)$$

generating simultaneous rotations of the pairs (ϕ_2^2, ϕ_2^3) , (p^2_2, p^2_3) resp. (ϕ_3^2, ϕ_3^3) , (p^3_2, p^3_3) . Thus the norms of these vectors and the scalar products between them are gauge invariant. The Gauss constraint allows us to get rid of e.g. the pair (ϕ_3^2, p^3_2) by fixing the gauge in the following way: we rotate the connection components such that $\phi_3^2 = 0$. Because the length $\|\phi_3\| = \sqrt{(\phi_3^2)^2 + (\phi_3^3)^2}$ is preserved we know that $\phi_3^3 \neq 0$. The Gauss constraint then implies that $p^3_2 = (\phi_2^2 p^2_3 - \phi_2^3 p^2_2) / \phi_3^3$. This gauge fixing reduces the degrees of freedom by two units.

The diffeomorphism constraint is given by Equation (26) and since $F_{ab}^i = \epsilon^i_{jk} A_a^j A_b^k$ ($\partial_a A_b^i = 0$ thanks to homogeneity) we find that

$$H_a = \epsilon^i_{jk} A_a^j A_b^k E_i^b \propto A_a^i G_i. \quad (31)$$

The gauge fixing we just performed ensures that the diffeomorphism constraint also vanishes.

3.5. Canonical Transformation

In this subsection we introduce a set of new variables which will greatly simplify the analysis of the kinematical Hilbert space. We first perform a canonical transformation on the unreduced phase space:

$$\begin{aligned} Q_1 &= \phi_1^1, & P^1 &= p^1_1, \\ Q_2 &= \sqrt{(\phi_2^2)^2 + (\phi_2^3)^2}, & P^2 &= \frac{p^2_2 \phi_2^2 + p^2_3 \phi_2^3}{\sqrt{(\phi_2^2)^2 + (\phi_2^3)^2}}, \\ Q_3 &= \sqrt{(\phi_3^2)^2 + (\phi_3^3)^2}, & P^3 &= \frac{p^3_2 \phi_3^2 + p^3_3 \phi_3^3}{\sqrt{(\phi_3^2)^2 + (\phi_3^3)^2}}, \\ \theta_1 &= \arccos \left(\frac{\phi_2^2}{\sqrt{(\phi_2^2)^2 + (\phi_2^3)^2}} \right), & P_{\theta_1} &= p^2_3 \phi_2^2 - p^2_2 \phi_2^3 \end{aligned} \quad (32)$$

$$\theta_2 = \arccos_k \left(\frac{\phi_3^3}{\sqrt{(\phi_3^2)^2 + (\phi_3^3)^2}} \right), \quad P_{\theta_2} = -p^3_3 \phi_3^2 + p^3_2 \phi_3^3$$

such that the variables are mutually conjugate:

$$\{Q_I, P^J\} = \frac{\kappa\gamma}{V_0} \delta_I^J, \quad \{\theta_\alpha, P_{\theta_\beta}\} = \frac{\kappa\gamma}{V_0} \delta_{\alpha,\beta}.$$

We choose the convention that the diagonal limit can be recovered by setting $\theta_1 = \theta_2 = 0$. The inverse of this canonical transformation will be important in the sequel and is given by:

$$\begin{aligned} \phi_2^2 &= Q_2 \cos(\theta_1), & \phi_2^3 &= Q_2 \sin(\theta_1), \\ p^2_2 &= P^2 \cos(\theta_1) - \frac{P_{\theta_1} \sin(\theta_1)}{Q_2}, & p^2_3 &= \frac{P_{\theta_1} \cos(\theta_1)}{Q_2} + P^2 \sin(\theta_1), \\ \phi_3^2 &= Q_3 \sin(\theta_2), & \phi_3^3 &= Q_3 \cos(\theta_2), \\ p^3_2 &= P^3 \sin(\theta_2) + \frac{P_{\theta_2} \cos(\theta_2)}{Q_3}, & p^3_3 &= -\frac{P_{\theta_2} \sin(\theta_2)}{Q_3} + P^3 \cos(\theta_2). \end{aligned} \quad (33)$$

It is important to note that $Q_2, Q_3 \in \mathbb{R}_+$ and $\theta_1, \theta_2 \in [k\pi, (k+1)\pi]$ where we restrict the values of k to be either $k = 0$ if $\text{sgn}(\phi_2^3) > 0$ or $k = 1$ if $\text{sgn}(\phi_2^3) < 0$. If $\text{sgn}(\phi_2^3) = 0$ then we have the case $k = 0$ if $\text{sgn}(\phi_2^2) > 0$ or $k = 1$ if $\text{sgn}(\phi_2^2) < 0$. The function $\arccos_1(x)$ is related to the principal value via $\arccos_1(x) = 2\pi - \arccos(x)$. With this convention we can recover Equation (32) unambiguously from Equation (33).

The Hamiltonian constraint (24) is given in terms of the new variables by

$$\begin{aligned} C_{\text{grav}} &= \frac{(2\kappa\gamma^2)^{-1}}{\sqrt{\left| \frac{P^1 [\cos(\theta_1 + \theta_2) (P_{\theta_1} P_{\theta_2} - P^2 P^3 Q_2 Q_3) + (P^2 P_{\theta_2} Q_2 + P^3 P_{\theta_1} Q_3) \sin(\theta_1 + \theta_2)]}{Q_2 Q_3} \right|}} \times \\ &\times \left\{ 2P^1 Q_1 \left[\cos(2\theta_2) P_{\theta_2} + P^2 Q_2 (\sin(2\theta_1) - 1) + P^3 Q_3 (\sin(2\theta_2) - 1) \right] \right. \\ &\quad \left. + P^2 Q_2 \left[P_{\theta_2} \sin(2(\theta_1 + \theta_2)) - 2 \cos^2(\theta_1 + \theta_2) P^3 Q_3 \right] \right. \\ &\quad \left. + P_{\theta_1} \left[2 \cos^2(\theta_1 + \theta_2) P_{\theta_2} + 2 \cos(2\theta_1) P^1 Q_1 + P^3 P_{\theta_3} \sin(2(\theta_1 + \theta_2)) \right] \right\} \end{aligned} \quad (34)$$

Using this Hamiltonian we can compute the time evolution of the basic variables Q_i , θ_α , P^i and P_{θ_α} (see Figure 5). We choose the initial conditions so that they correspond to the values of the old variables (see caption of Figure 4). By doing so we are able to check whether the solutions to Equation (24) are equivalent to the solutions to Equation (34) by performing the canonical transformation (32). The different solutions do indeed match up to a very good accuracy.

The only nontrivial Gauss constraint (30) is then given by

$$G_1 = P_{\theta_1} - P_{\theta_2}, \quad (35)$$

which vanishes only when $P_{\theta_2} = P_{\theta_1}$. We are free to fix the gauge by setting $\theta_2 = 0$. The same result can be obtained from the gauge fixing performed in Section 3.4.2 so that

$$Q_3 = \phi_3^3, \quad P^3 = p^3_3, \quad \theta_2 = 0 \quad \text{and} \quad P_{\theta_2} = P_{\theta_1}.$$

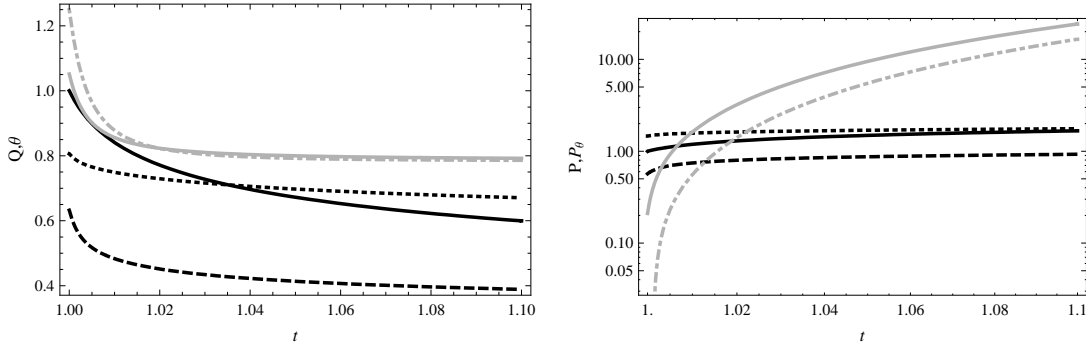


Figure 5. Solutions corresponding to the Hamiltonian (34) coupled to a massless scalar field with vanishing potential. *Left panel:* the black thick solid shows the evolution of Q_1 , Q_2 is the black dashed line, Q_3 the dotted line, θ_1 the gray dashdotted one and θ_2 the solid gray one. *Right panel:* the black thick solid shows the evolution of P^1 , P^2 is the black dashed line, P^3 the dotted line, P_{θ_1} the gray dashdotted one and P_{θ_2} the solid gray one. In both cases the initial conditions are $Q_1 = 1$, $Q_2 = 0.63$, $Q_3 = 0.81$, $\theta_1 = 1.25$, $\theta_2 = 1.05$, $P^1 = 1$, $P^2 = 0.57$, $P^3 = 1.46$, $P_{\theta_1} = -0.08$, $P_{\theta_2} = 0.21$, $\phi = 0.01$ and $p_\phi = 8.1$. The time t parametrizes the coordinate time in natural units ($c = \kappa = \hbar = 1$).

The symplectic structure of the reduced 8-dimensional phase space is given by

$$\Omega = \frac{V_0}{\kappa\gamma} (dQ_1 \wedge dP^1 + dQ_2 \wedge dP^2 + dQ_3 \wedge dP^3 + d\theta_1 \wedge dP_{\theta_1}).$$

4. Kinematical Hilbert Space

4.1. Holonomies

In the last section we parametrized the classical phase space and gave the Hamiltonian in terms of the Ashtekar variables. To quantize the theory we have to select a set of elementary observables which have unambiguous operator analogs. In order to do so we have first to find elementary variables of the 5-dimensional configuration space.

According to [10] the configuration space on the covering space $\tilde{\Sigma}$ is given by Higgs fields in a single point x_0 which is the only point in the reduced manifold $\tilde{\Sigma}/\tilde{S}$. In quantum theory these fields are represented as point holonomies associated to the point x_0 [33]. On \tilde{S} we can take the three edges ξ_1 , ξ_2 and ξ_3 in order to regularize the point holonomies. However, we would like to apply this construction to a closed manifold. First we note that $\Sigma \cong \tilde{S}/\Gamma$ such that two elements $g, g' \in \tilde{S}$ are equivalent if there is an element $\gamma \in \Gamma$ such that $g' = g + \gamma$. We can thus restrict the regularization of the point holonomies to the three edges X_1 , X_2 and X_3 meeting at x_0 without losing information. Our elementary configuration variables are then the holonomies along straight lines $\gamma_I : [0, 1] \rightarrow \Sigma$ defined by the connection $A \propto (\phi_I^i)$ [10, 11, 12, 13, 14]. Now the holonomies along X_1 , X_2 resp. X_3 are given by

$$h_1^{(\lambda_1)} = \exp(\lambda_1 \phi_1^1 \tau_1) = \cos(\lambda_1 \phi_1^1) + 2\tau_1 \sin(\lambda_1 \phi_1^1)$$

$$\begin{aligned} h_2^{(\lambda_2)} &= \exp(\lambda_2 \phi_2^2 \tau_2 + \lambda_2 \phi_2^3 \tau_3) \\ h_3^{(\lambda_3)} &= \exp(\lambda_3 \phi_3^2 \tau_2 + \lambda_3 \phi_3^3 \tau_3), \end{aligned} \quad (36)$$

where $\lambda_I \in (-\infty, \infty)$ and $\lambda_I L_I$ is the length of the edge I with respect to the spatial metric h_{ab} . The auxiliary Hilbert space is then generated by spin networks associated with graphs consisting of the three edges γ_I meeting at the vertex x_0 .

In open Bianchi type I models the gauge invariant information of the connection can be separated from the gauge degrees of freedom via the relation $\phi_I^i = c_{(I)} \Lambda_I^i$ with $\Lambda \in SO(3)$ so that the holonomies become simple trigonometric functions. In our case the situation is more complicated because the holonomies h_2 and h_3 cannot be reduced to such functions since

$$h_\alpha^{(\lambda_\alpha)} = \cos\left(\lambda_\alpha \|\vec{\phi}_\alpha\|/2\right) + 2 \frac{\phi_\alpha^i \tau_i}{\|\vec{\phi}_\alpha\|} \sin\left(\lambda_\alpha \|\vec{\phi}_\alpha\|/2\right) \quad (\text{no summation}),$$

where $\alpha = 2, 3$ and

$$\|\vec{\phi}_\alpha\| := \sqrt{\sum_i (\phi_\alpha^i)^2}.$$

The problem is that this expression cannot be used in this form since there is no well defined operator $\hat{\phi}_I^i$ on the kinematical Hilbert space. Using the canonical transformation (32) we can re-express the holonomies such that

$$\begin{aligned} h_1^{(\lambda_1)} &= \cos(\lambda_1 Q_1/2) + 2\tau_1 \sin(\lambda_1 Q_1/2), \\ h_2^{(\lambda_2)} &= \cos(\lambda_2 Q_2/2) + 2(\tau_2 \cos \theta_1 + \tau_3 \sin \theta_1) \sin(\lambda_2 Q_2/2), \\ h_3^{(\lambda_3)} &= \cos(\lambda_3 Q_3/2) + 2(\tau_2 \sin \theta_2 + \tau_3 \cos \theta_2) \sin(\lambda_3 Q_3/2). \end{aligned} \quad (37)$$

Since $\lambda_I \in \mathbb{R}$ matrix elements of the exponentials of Q_1 , Q_2 and Q_3 form a C^* -algebra of almost periodic functions. On the other hand the variables $\theta_{1,2}$ are periodic angles such that only strictly periodic functions $\exp(ik_\alpha \theta_\alpha) \in U(1)$ with $k_\alpha \in \mathbb{Z}$ are allowed. Thus, any function generated by this set can be written as

$$\begin{aligned} g(Q_1, Q_2, Q_3, \theta_1, \theta_2) &= \sum_{\lambda_1, \lambda_2, \lambda_3, k_1, k_2} \xi_{\lambda_1, \lambda_2, \lambda_3, k_1, k_2} \times \\ &\times \exp\left(\frac{1}{2}i\lambda_1 Q_1 + \frac{1}{2}i\lambda_2 Q_2 + \frac{1}{2}i\lambda_3 Q_3 + ik_1 \theta_1 + ik_2 \theta_2\right) \end{aligned} \quad (38)$$

with coefficients $\xi_{\lambda_1, \lambda_2, \lambda_3, k_1, k_2} \in \mathbb{C}$, generating the C^* -algebra \mathcal{A}_S . Note that this function is almost periodic in Q_1, Q_2 and Q_3 and strictly periodic in θ_1 and θ_2 . The spectrum of the algebra of the almost periodic functions is called the Bohr compactification $\bar{\mathbb{R}}_B := \Delta(\text{Cyl}_S)$ of the real line and can be seen as the space of generalized connections [13, 34]. Thus the functions (38) provide us a complete set of continuous functions on $\bar{\mathbb{R}}_B \times \bar{\mathbb{R}}_B \times \bar{\mathbb{R}}_B \times S^1 \times S^1$. Moreover the Gel'fand theory guarantees that the space $\bar{\mathbb{R}}_B$ is compact and Hausdorff [35] with a unique normalized Haar measure $d\mu(c)$ such that

$$\int f(c) d\mu(c) := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(c) dc.$$

A Cauchy completion leads to a Hilbert space \mathcal{H}^S defined by the tensor product $\mathcal{H}^S = \mathcal{H}_B^{\otimes 3} \otimes \mathcal{H}_{S^1}^{\otimes 2}$ with the Hilbert spaces $\mathcal{H}_B = L^2(\bar{\mathbb{R}}_B, d\mu(c))$ and $\mathcal{H}_{S^1} = L^2(S^1, d\phi)$ of square integrable functions on $\bar{\mathbb{R}}_B$ and the circle respectively, where $d\phi$ is the Haar measure for S^1 . An orthonormal basis for \mathcal{H}_B is given by the almost periodic functions $\langle Q_I | \mu_I \rangle = \exp(i\mu_I Q_I/2)$ (no summation) with $\mu_I \in \mathbb{R}$ with $\langle \mu_I | \mu'_I \rangle = \delta_{\mu_I, \mu'_I}$. Analogously a basis for \mathcal{H}_{S^1} is given by the strictly periodic functions $\langle \theta_\alpha | k_\alpha \rangle = \exp(ik_\alpha \theta_\alpha)$ with $\langle k_\alpha | k'_\alpha \rangle = \delta_{k_\alpha, k'_\alpha}$.

We choose a representation where the configuration variables, now promoted to operators, act by multiplication via:

$$(\hat{g}_1 \hat{g}_2)(\vec{Q}, \vec{\theta}) = g_1(\vec{Q}, \vec{\theta}) g_2(\vec{Q}, \vec{\theta}).$$

The momentum operators act by derivation in the following way:

$$\hat{P}^I = -i\gamma l_{\text{Pl}}^2 \frac{\partial}{\partial Q_I}, \quad \hat{P}_{\theta_\alpha} = -i\gamma l_{\text{Pl}}^2 \frac{\partial}{\partial \theta_\alpha}. \quad (39)$$

The eigenstates of all momentum operators are given by

$$\begin{aligned} |\vec{\mu}, \vec{k}\rangle &:= |\mu_1, \mu_2, \mu_3, k_1, k_2\rangle \\ &:= |\mu_1\rangle \otimes |\mu_2\rangle \otimes |\mu_3\rangle \otimes |k_1\rangle \otimes |k_2\rangle \end{aligned}$$

with

$$\hat{P}^I |\vec{\mu}, \vec{k}\rangle = \gamma l_{\text{Pl}}^2 \mu_I |\vec{\mu}, \vec{k}\rangle, \quad \hat{P}_{\theta_\alpha} |\vec{\mu}, \vec{k}\rangle = \frac{\gamma l_{\text{Pl}}^2}{2} k_\alpha |\vec{\mu}, \vec{k}\rangle. \quad (40)$$

The simple form of the momentum operators (39) may suggest that the Hilbert space of LQC on a torus is simply expanded from $L^2(\bar{\mathbb{R}}_B^3)$ to $L^2(\bar{\mathbb{R}}_B^3) \times L^2(U(1)^2)$. However the situation is far more complicated because the important variables for the Gauss and Hamiltonian constraints are not the new momenta P^I and P_{θ_α} but the components p^I_i of the triad. In terms of the new canonical variables they are complicated functions of both the configuration and momentum variables, as can be seen from Equation (33). These expressions cannot be quantized directly since the operators $\hat{Q}_{2,3}$ fail to be well defined on the Hilbert space. The solution is to consider the momentum operators of the full theory given by a sum of left and right invariant vector fields. In [11] the same strategy was used to show that the triad components p^I_i act by derivation. In our case the situation is more complicated since the triad components contain both configuration and momentum variables. The triad operators act on functions in \mathcal{H}^S and are given by

$$\hat{p}^I_i = -i \frac{\gamma l_{\text{Pl}}^2}{2} \left(X_i^{(R)}(h_I) + X_i^{(L)}(h_I) \right), \quad (41)$$

where $X_i^{(R)}(h_I)$ and $X_i^{(L)}(h_I)$ are the right and left invariant vector fields acting on the copy of $SU(2)$ associated with the edge e_I of length 1 and are given by

$$X_i^{(R)}(h_I) = \text{tr} \left[(\tau_i h_I)^T \frac{\partial}{\partial h_I} \right], \quad X_i^{(L)}(h_I) = \text{tr} \left[(h_I \tau_i)^T \frac{\partial}{\partial h_I} \right].$$

Applying the operators \hat{p}^2_2 and \hat{p}^2_3 on the function $\text{tr}(h_2)$ we get

$$\begin{aligned}\hat{p}^2_2 \text{tr}(h_2) &= 2\hat{p}^2_2 \cos(\lambda_2 Q_2/2) = i\gamma l_{\text{Pl}}^2 \lambda_2 \sin(\lambda_2 Q_2/2) \cos(\theta_1), \\ \hat{p}^2_3 \text{tr}(h_2) &= 2\hat{p}^2_3 \cos(\lambda_2 Q_2/2) = i\gamma l_{\text{Pl}}^2 \lambda_2 \sin(\lambda_2 Q_2/2) \sin(\theta_1).\end{aligned}\quad (42)$$

We see that the usual expressions for an open topology can be recovered by simply setting $\theta_1 = 0$. Applying these operators once again we get the expressions:

$$(\hat{p}^2_2)^2 \text{tr}(h_2) = \frac{1}{2} \gamma^2 l_{\text{Pl}}^4 \lambda_2^2 \cos(\lambda_2 Q_2/2) = (\hat{p}^2_3)^2 \text{tr}(h_2),$$

which means that $\cos(\lambda_2 Q_2/2)$ is an eigenfunction of both $(\hat{p}^2_2)^2$ and $(\hat{p}^2_3)^2$ with eigenvalue $\gamma^2 l_{\text{Pl}}^4 / 2\lambda_2^2$. On the other hand we have

$$\hat{p}^2_2 \hat{p}^2_3 \text{tr}(h_2) = \hat{p}^2_3 \hat{p}^2_2 \text{tr}(h_2) = 0.$$

4.2. Quantization: 1. Possibility

As previously mentioned we cannot directly quantize the expressions (33) because \hat{Q}_I does not exist as multiplication operator on \mathcal{H}^S . In a loop quantization only holonomies of the connections are represented as well-defined operators on \mathcal{H}^S . Thus we replace every configuration variable Q_I in Equation (33) by $\sin(\delta_I Q_I/2)/\delta_I$ [36], where $\delta_I \in \mathbb{R} \setminus \{0\}$ plays the role of a regulator, and compare it with the results just obtained in terms of left and right invariant vector fields. For later purpose we order the operators in a symmetrical way get the following operators acting on functions of \mathcal{H}^S :

$$\begin{aligned}\hat{\phi}_2^2 &= \frac{\sin(\delta_2 Q_2)}{\delta_2} \cos \theta_1, & \hat{p}^2_2 &= \cos \theta_1 \hat{P}^2 - \frac{\delta_2 \sqrt{\sin \theta_1}}{\sin(\delta_2 Q_2)} \hat{P}_{\theta_1} \sqrt{\sin \theta_1}, \\ \hat{\phi}_2^3 &= \frac{\sin(\delta_2 Q_2)}{\delta_2} \sin \theta_1, & \hat{p}^2_3 &= \sin \theta_1 \hat{P}^2 + \frac{\delta_2 \sqrt{\cos \theta_1}}{\sin(\delta_2 Q_2)} \hat{P}_{\theta_1} \sqrt{\cos \theta_1}, \\ \hat{\phi}_3^2 &= \frac{\sin(\delta_3 Q_3)}{\delta_3} \sin \theta_2, & \hat{p}^3_2 &= \sin \theta_2 \hat{P}^3 + \frac{\delta_3 \sqrt{\cos \theta_2}}{\sin(\delta_3 Q_3)} \hat{P}_{\theta_2} \sqrt{\cos \theta_2}, \\ \hat{\phi}_3^3 &= \frac{\sin(\delta_3 Q_3)}{\delta_3} \cos \theta_2, & \hat{p}^3_3 &= \cos \theta_2 \hat{P}^3 - \frac{\delta_3 \sqrt{\sin \theta_2}}{\sin(\delta_3 Q_3)} \hat{P}_{\theta_2} \sqrt{\sin \theta_2}.\end{aligned}\quad (43)$$

Applying e.g. the operator \hat{p}^2_2 on $\cos(\lambda_2 Q_2/2)$ with the definitions (40) we see that we obtain the same result as Equation (42) for $\delta = 1$. This is not surprising in view of the fact that we defined the operator \hat{p}^I_i in Equation (41) with holonomies along edges e_I of length 1.

This substitution is problematic since the configuration variables $Q_{2,3}$ are by definition positive (see Equation (32)). Therefore, for $Q_{2,3} \rightarrow \sin(\delta_{2,3} Q_{2,3})/\delta_{2,3}$ to be valid we restrict the analysis to the domain $0 < Q_{2,3} < \pi$. In the diagonal case the situation is less problematic because the configuration variable c is arbitrary such that $\sin(\delta c)$ is also allowed to be negative.

Classically, since the change of variables (33) is a canonical transformation the symplectic structure is conserved, i.e. the Poisson bracket between p^2_2 and p^2_3 vanishes:

$$\{p^2_2, p^2_3\}_{Q,P} = 0$$

A quantization of the above expression is obtained with the substitution $\{, \} \rightarrow -i[,]\hbar$ such that the commutator between \hat{p}^2_2 and \hat{p}^2_3 should also vanish. However, the consequence of the substitution of $1/Q_I$ by $\delta_I/\sin(\delta_I Q_I)$ is that commutator between these two variables doesn't vanish anymore:

$$[\hat{p}^2_2, \hat{p}^2_3] f(Q_2, \theta_1) = -\gamma^2 l_{\text{Pl}}^4 \delta_2 \frac{\cos(\delta_2 Q_2) - \delta_2}{\sin^2(\delta_2 Q_2)} \frac{\partial f}{\partial \theta_1} \quad (44)$$

Formally we can recover the classical limit by taking the limit

$$\lim_{\delta_2 \rightarrow 0} [\hat{p}^2_2, \hat{p}^2_3] f(Q_2, \theta_1) = 0,$$

which however fails to exist on \mathcal{H}^S .

The operators \hat{p}^I_i are partial differential operators with periodic coefficients in both θ and Q . In spherically symmetric quantum geometry a similar situation arises when considering the quantization of a nondiagonal triad component [36]. However the expression of this component reduces to a Hamiltonian whose eigenvalues are discrete. In our case the situation is more complicated.

4.2.1. Quantization of p^2_2 In order to find eigenfunctions of the triad operators let us consider an operator of the form

$$\hat{A}_\delta := -i \cos \theta \frac{\partial}{\partial Q} + i \frac{\delta \sqrt{\sin(\theta)}}{\sin(\delta Q)} \frac{\partial}{\partial \theta} \sqrt{\sin \theta}.$$

A substitution $\xi = \delta Q$ shows that $\hat{A}_\delta = \delta \hat{A}_1 \equiv \delta \hat{A}$ so that it is sufficient to determine the spectrum for $\delta = 1$. This operator is symmetric on $\mathcal{H}_A := L^2(\bar{\mathbb{R}}_B, d\mu_B) \otimes L^2(U(1))$:

$$\langle f, \hat{A}g \rangle = \langle \hat{A}f, g \rangle, \quad \forall f, g \in \mathcal{D}(\hat{A}),$$

where $\mathcal{D}(\hat{A}) \subset \mathcal{H}_A$ is the domain of \hat{A} . The eigenfunctions of \hat{A} are obtained by solving $\hat{A}f_\lambda(\xi, \theta) = \lambda f_\lambda(\xi, \theta)$, i.e.

$$-i \cos \theta \frac{\partial f_\lambda(\xi, \theta)}{\partial \xi} + i \frac{\sin \theta}{\sin \xi} \frac{\partial f_\lambda(\xi, \theta)}{\partial \theta} + \frac{i \cos \theta}{2 \sin \xi} f_\lambda(\xi, \theta) = \lambda f_\lambda(\xi, \theta), \quad (45)$$

where we constrain ξ to be in the interval $[0, \pi]$ in order to avoid negative values of $\sin \xi$. We look for a solution of the form $w = w(\xi, \theta)$ [37] satisfying

$$-i \cos \theta \frac{\partial w}{\partial \xi} + i \frac{\sin \theta}{\sin \xi} \frac{\partial w}{\partial \theta} = \left(\lambda - \frac{i \cos \theta}{2 \sin \xi} \right) f_\lambda \frac{\partial w}{\partial f_\lambda}$$

such that the characteristic functions are given by

$$\dot{\xi} = -i \cos \theta(t), \quad \dot{\theta} = i \frac{\sin \theta(t)}{\sin \xi(t)} \quad \text{and} \quad \dot{f}_\lambda = \left(\lambda - \frac{i \cos \theta(t)}{2 \sin \xi(t)} \right) f_\lambda(t), \quad (46)$$

where the dot is the time derivative. Combining the first two equations gives after integration

$$\sin \theta \tan \frac{\xi}{2} = C_1, \quad (47)$$

meaning that every C^1 -function $\Omega_1(\sin \theta \tan(\xi/2))$ solves the left-hand side of Equation (45). In order to solve Equation (45) for $\lambda \neq 0$ we first note that

$$\cos \theta(t) = \cos(\arcsin(C_1 \cot(\xi/2))) = \sqrt{1 - C_1^2 \cot^2(\xi/2)} \equiv i\dot{\xi}. \quad (48)$$

An integration of this equation gives the result

$$t = -i \frac{\sqrt{2}b \log(\sqrt{2}a \cos(\xi/2) + b)}{a\sqrt{1 - C_1^2 \cot^2(\xi/2)} |\sin(\xi/2)|}, \quad (49)$$

where

$$a = \sqrt{1 + C_1^2} \quad \text{and} \quad b = \sqrt{-1 + C_1^2 + \cos \xi(1 + C_1^2)}.$$

The last characteristic equation in (46) can be written as

$$\dot{f}_\lambda = \frac{\partial f_\lambda}{\partial \xi} \dot{\xi} = \left(\lambda - \frac{i \cos \theta}{2 \sin \xi} \right) f_\lambda$$

such that

$$\frac{\partial f_\lambda}{\partial \xi} = \left(i \frac{\lambda}{\cos \theta} + \frac{1}{2 \sin \xi} \right) f_\lambda.$$

Equation (48) can be inserted into the last equation such that after an integration we get the result

$$\log f_\lambda = \lambda t + \log \left(\sqrt{\tan(\xi/2)} \right) + C,$$

where t is given by Equation (49) and C is an integration constant. The final solution to the PDE (45) is thus given by

$$f_\lambda(\xi, \theta) = \mathcal{N}_1 \sqrt{\tan(\xi/2)} \times \\ \times \left(\sqrt{2} \cos(\xi/2) \alpha_1 + \beta_1 \right)^{-i \frac{\sqrt{2} \lambda \beta_1}{\alpha_1 |\sin(\xi/2)| \cos \theta}} \Omega_1(\sin \theta \tan(\xi/2)), \quad (50)$$

where

$$\alpha_1(\xi, \theta) = \sqrt{1 + \sin^2 \theta \tan^2(\xi/2)} \quad \text{and} \\ \beta_1(\xi, \theta) = \sqrt{-1 + \cos \xi + (1 + \cos \xi) \tan^2(\xi/2) \sin^2 \theta}.$$

The C^1 -function $\Omega_1(\sin \theta \tan(\xi/2))$ can be determined by e.g. boundary conditions. For simplicity we set $\Omega_1(\sin \theta \tan(\xi/2)) \equiv 1$ subsequently. As a cross-check we see that the first line of Equation (50) solves

$$-i \cos \theta \frac{\partial \sqrt{\tan(\xi/2)}}{\partial \xi} + \frac{i \cos \theta}{2 \sin \xi} \sqrt{\tan(\xi/2)} = 0$$

and the second one the eigenvalue problem of the operator \hat{A} . The scalar product on \mathcal{H}_A is given by

$$\langle f_\lambda, f_{\lambda'} \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\xi \int_0^{2\pi} d\theta \bar{f}_\lambda f_{\lambda'}. \quad (51)$$

The integral of $|\sqrt{\tan(\xi/2)}|^2$ over one period is not finite and since the second line of Equation (50) never vanishes the eigenfunctions f_λ are not normalizable in \mathcal{H}_A . We

could choose the function $\Omega_1 \propto (\tan(\xi/2))^{-1/2}$ but we would automatically get the factor $(\sin \theta)^{-1/2}$ which is also not normalizable. The surprising implication is that the spectrum of \hat{A} is continuous. Note that the function α_1 is always real while β_1 is always purely imaginary ($\lim_{\xi \rightarrow \pi/2} \beta_1 = i \cos \theta$). The exponent of f_λ is thus always real, implying that f_λ is uniquely determined.

4.2.2. Self-adjointness of \hat{A} In the previous section we constructed a symmetric operator \hat{A} with respect to the scalar product of \mathcal{H}_A , i.e. $\hat{A} = \hat{A}^+$ with domain $\mathcal{D}(\hat{A}) \subset \mathcal{D}(\hat{A}^+)$. In this subsection we give a possible domain for \hat{A} and check if there exists a self-adjoint extension of \hat{A} .

Definition 5. *In analogy with [38, 39] define the space $\text{CAP}(\mathbb{R})$ of the (uniform) almost periodic functions* such that its completion is the Hilbert space $L^2(\bar{\mathbb{R}}_B)$. The Sobolev space $H^1(\bar{\mathbb{R}}_B)$ is given by the completion of the space of trigonometric polynomials $\text{Trig}(\mathbb{R})$ in the Sobolev norm $\|f\|_{H^1}^2 = \|f\|_{L^2(\bar{\mathbb{R}}_B)}^2 + \|f'\|_{L^2(\bar{\mathbb{R}}_B)}^2$, i.e. $H^1(\bar{\mathbb{R}}_B)$ consists of all almost periodic functions $f \in \text{CAP}(\mathbb{R})$ such that $f' \in \text{CAP}(\mathbb{R})$.*

Let the differential operator $\hat{p} := -i \frac{d}{d\xi}$ on $L^2(\bar{\mathbb{R}}_B)$ have the domain of definition $\text{Trig}(\mathbb{R})$. Then its closure has the domain $H^1(\bar{\mathbb{R}}_B)$. The adjoint operator to \hat{p} on $L^2(\bar{\mathbb{R}}_B)$ has also the domain $H^1(\bar{\mathbb{R}}_B)$ and coincides with \hat{p}^+ on it. Since $\hat{p} = \hat{p}^+$, \hat{p} is essentially self-adjoint on $\text{Trig}(\mathbb{R})$ [38, 39].

Since every almost periodic function $f(x)$ is bounded a necessary condition for the inverse $f^{-1}(x)$ to be almost periodic is that $\min_x |f(x)| \neq 0$. It follows that $\sin^{-1} \xi$ is not an almost periodic function. We thus define the domain

$$\mathcal{D}(\hat{A}) := \{\varphi \in H^1(\bar{\mathbb{R}}_B) \otimes H^1(U(1)) \mid \varphi(k\pi, \theta) = 0 = \varphi'(k\pi, \theta), k \in \mathbb{Z}\}, \quad (52)$$

which, according to [41, 42], is dense. Any function $\varphi \in \mathcal{D}(\hat{A})$ removes the pole caused by $\sin^{-1} \xi$, i.e. we require that $\lim_{\xi \rightarrow k\pi} \varphi(\xi)(\sin \xi)^{-1} = 0$ and $\lim_{\xi \rightarrow k\pi} \varphi'(\xi)(\sin \xi)^{-1} = 0$. $k \in \mathbb{Z}$. On the other hand, thanks to $\sin \theta$ in front of the differential operator $i\partial/\partial\theta$, the boundary term of an integration by part is automatically annihilated so that no boundary conditions on θ have to be imposed. Moreover the deficiency indices n_\pm for \hat{A} are defined by

$$n_\pm := \dim \ker(\hat{A}^+ \mp i).$$

The solutions to this equation do not lie in $\mathcal{D}(\hat{A}^+)$ such that $n_\pm = 0$. It follows that the operator \hat{A} is essentially self-adjoint.

4.2.3. Quantization of p^2_3 The eigenfunctions of \hat{p}^2_3 can be obtained by applying the same procedure on the symmetrized operator

$$\hat{B} := -i \sin \theta \frac{\partial}{\partial \xi} - i \frac{\sqrt{\cos \theta}}{\sin \xi} \frac{\partial}{\partial \theta} \sqrt{\cos \theta}$$

* An almost periodic function $f(x)$ is uniformly continuous for $x \in \mathbb{R}$ and bounded [40].

The eigenfunctions $g_\lambda(\xi, \theta)$ are given by

$$g_\lambda(\xi, \theta) = \frac{\mathcal{N}_2}{\sqrt{\tan(\xi/2)}} \left(\sqrt{2} \cos(\xi/2) \alpha_2 + \beta_2 \right)^{-i \frac{\sqrt{2}\lambda\beta_2}{\alpha_2 |\sin(\xi/2)| \sin \theta}} \Omega_2(\cos \theta \tan(\xi/2)), \quad (53)$$

where

$$\alpha_2 = \sqrt{1 - \frac{\cot^2(\xi/2)}{\cos^2 \theta}} \quad \text{and}$$

$$\beta_2 = \sqrt{-1 + \cos \xi + \frac{\cot^2(\xi/2)}{\cos^2 \theta} (\cos \xi - 1)}$$

and Ω_2 is any C^1 -function that can be determined by boundary conditions. While the function β_2 is always purely imaginary the function α_2 is only real when $\cot^2 \xi/2 < \cos^2 \theta$. This means that g_λ is not uniquely determined. We can write g_λ as

$$g_\lambda(\xi, \theta) = \frac{k_2}{\sqrt{\tan(\xi/2)}} e^{F_1(\xi, \theta) \ln F_2(\xi, \theta)}$$

with the logarithm is defined by $\ln F_2 = \text{Ln} F_2 + 2\pi i n$, where $n \in \mathbb{Z}$ and Ln is the principal value of the logarithm. Inserting this solution into the eigenvalue problem $\hat{B}g_\lambda = \lambda g_\lambda$ it can be shown that there is only a solution for $n = 0$. The eigenfunctions g_λ are not normalizable since the integral of $1/|\tan(\xi/2)|$ over one period is not finite.

As for f_λ we are led to the conclusion that the spectrum of \hat{B} is continuous. We can construct a dense subspace along the lines described in Section 4.2.2, the only difference being that g_λ has poles at $\xi = 2k\pi$ and $\theta = (2k + 1)\pi/2$ whereas f_λ has poles at $\xi = (2k + 1)\pi$, $k \in \mathbb{Z}$.

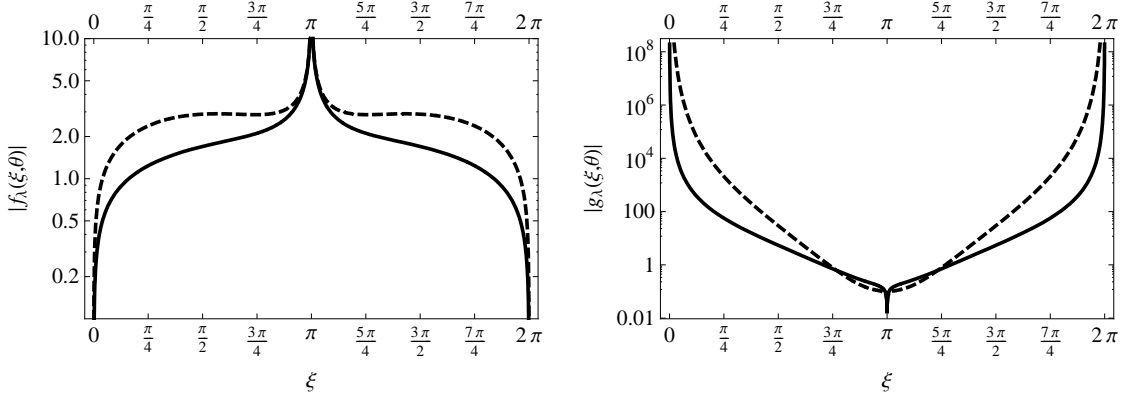


Figure 6. Absolute value of the eigenfunctions $f_\lambda(\xi, \theta)$ (left panel) and $g_\lambda(\xi, \theta)$ (right panel). The black thick line is the eigenfunction for $\lambda = 1$, $\theta = 1$ and the black dashed line for $\lambda = 2$, $\theta = 1$.

4.3. Quantization: 2. Possibility

In order to quantize the triad components \hat{p}^I_i we replaced the configuration variables Q_I with $\sin(\delta_I Q_I)/\delta_I$. The question we may ask is to what extent this substitution

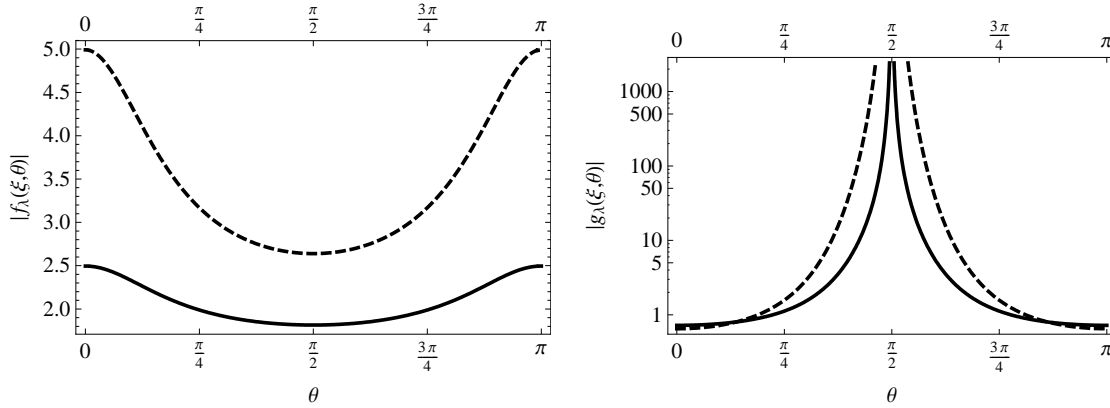


Figure 7. Absolute value of the eigenfunctions $f_\lambda(\xi, \theta)$ (left panel) and $g_\lambda(\xi, \theta)$ (right panel). The black thick line is the eigenfunction for $\lambda = 1, \xi = 2$ and the black dashed line for $\lambda = 2, \xi = 2$.

changes the eigenfunctions. We define the symmetrized operator \hat{A}_2 quantized without the substitution of Q_I as

$$\hat{A}_2 = -i \cos \theta \frac{\partial}{\partial \xi} + i \frac{\sin \theta}{\xi} \frac{\partial}{\partial \theta} + \frac{i \cos \theta}{2 \xi}.$$

The solution to the eigenvalue problem $\hat{A}_2 f_\lambda(\xi, \theta) = \lambda f_\lambda(\xi, \theta)$ is given by

$$f_\lambda(\xi, \theta) = \exp(i\lambda\xi \cos \theta) \sqrt{\xi} \Gamma(\log(\xi \sin \theta)),$$

We see that the eigenfunctions are not almost periodic in ξ . However we can choose the function Γ such that $\sqrt{\xi}$ disappears, i.e. we set

$$\Gamma = \mathcal{N}_1 \exp\left(-\frac{1}{2} \log(\xi \sin \theta)\right),$$

where \mathcal{N}_1 is a constant, such that the eigenfunctions to \hat{A}_2 are given by

$$f_\lambda(\xi, \theta) = \mathcal{N}_1 \frac{\exp(i\lambda\xi \cos \theta)}{\sqrt{\sin \theta}}. \quad (54)$$

The above eigenfunction is almost periodic in ξ but fails to be normalizable on $L^2(\bar{\mathbb{R}}_B) \otimes L^2(U(1))$. As in the preceding section the spectrum of \hat{A}_2 is thus continuous. Note that the eigenfunction is constant in the non-diagonal limit $\theta \rightarrow \pi/2$.

Similarly the eigenfunctions of the symmetrized operator

$$\hat{B}_2 = -i \sin \theta \frac{\partial}{\partial \xi} - i \frac{\cos \theta}{\xi} \frac{\partial}{\partial \theta} + \frac{i \sin \theta}{2 \xi}$$

are given by

$$g_\lambda(\xi, \theta) = \mathcal{N}_2 \frac{\exp(-i\lambda\xi \sin \theta)}{\sqrt{\cos \theta}}. \quad (55)$$

The diagonal limit $\theta \rightarrow 0$ of g_λ is just a constant function such that \hat{p}^2_3 , i.e. the expectation value $\langle \hat{p}^2_2 \rangle$ measures the 'diagonality' of the torus and $\langle \hat{p}^2_3 \rangle$ its departure. Once again, the eigenfunctions g_λ fail to be normalizable on the Hilbert space such that the spectrum of \hat{B}_2 is continuous. Also note that contrary to the first method the commutator between both operators vanishes:

$$[\hat{A}_2, \hat{B}_2] = 0.$$

4.4. Volume Operator

The classical expression for the volume of V is given by

$$\mathcal{V}(V) = \int_V \sqrt{\left| \frac{1}{6} \epsilon_{abc} \epsilon_{ijk} E^{ai} E^{bj} E^{ck} \right|} d^3x.$$

Inserting the definition of the homogeneous densitized triad (17) we get:

$$\mathcal{V}(V) = \sqrt{\mathfrak{k} |p^1_1 (p^2_2 p^3_3 - p^2_3 p^3_2)|} \quad (56)$$

The factor \mathfrak{k} depends on the specific form of the torus and is equal to one if the torus is cubic such that we recover the usual expression in this limit (see e.g. Eq. (4.5) in [31]). Using the classical solution of the Gauss constraint we get the following expression for the physical volume of the torus:

$$\mathcal{V}(V) = \sqrt{\mathfrak{k} \left| p^1_1 \left[p^2_2 p^3_3 - p^2_3 \frac{\phi_2^2 p^2_3 - \phi_2^3 p^2_2}{\phi_3^3} \right] \right|}$$

or in terms of the new variables

$$\begin{aligned} \mathcal{V}(V) = \sqrt{\mathfrak{k} \left| \frac{P^1}{Q_2 Q_3} \right|} &\times \left| ((P_\Theta)^2 - P^2 P^3 Q_2 Q_3) \cos \Theta + \right. \\ &\left. + P_\Theta (P^2 Q_2 + P^3 Q_3) \sin \Theta \right|^{1/2}. \end{aligned} \quad (57)$$

4.4.1. Quantization of the Volume Operator according to 1. Method To perform a quantization of the volume operator we insert the definitions (43) into $\hat{\mathcal{V}}(V)$. Despite the fact that we know the eigenfunctions of the operators \hat{p}^I_i it is not straightforward to give the eigenfunctions of the volume operator $\hat{\mathcal{V}}$ because, as explained in Section 4.1, they do not necessarily commute. Thus the difficult task is to determine the spectrum of the operator

$$\begin{aligned} \hat{\mathfrak{V}} := \hat{p}^2_2 \hat{p}^3_3 - \hat{p}^3_2 \hat{p}^2_3 &= \frac{\cos \Theta}{\sin Q_2 \sin Q_3} \frac{\partial^2}{\partial \Theta^2} - \cos \Theta \frac{\partial^2}{\partial Q_2 \partial Q_3} \\ &+ \frac{\sin \Theta}{\sin Q_3} \frac{\partial^2}{\partial \Theta \partial Q_2} + \frac{\sin \Theta}{\sin Q_2} \frac{\partial^2}{\partial \Theta \partial Q_3}. \end{aligned} \quad (58)$$

However, this operator is not symmetric on \mathcal{H}^S . Let us define the symmetric operator

$$\hat{\mathfrak{V}}^S := \frac{1}{2} (\hat{\mathfrak{V}}^+ + \hat{\mathfrak{V}}).$$

A calculation shows that $\hat{\mathfrak{V}}^S$ is given by

$$\hat{\mathfrak{V}}^S = \hat{\mathfrak{V}} + \frac{1}{2} \left(-\frac{\cos \Theta}{\sin Q_2 \sin Q_3} - 2 \frac{\sin \Theta}{\sin Q_2 \sin Q_3} \frac{\partial}{\partial \Theta} + \frac{\cos \Theta}{\sin Q_3} \frac{\partial}{\partial Q_2} + \frac{\cos \Theta}{\sin Q_2} \frac{\partial}{\partial Q_3} \right)$$

This operator is rather complicated and no analytic solutions to the eigenvalue problem could be found.

4.4.2. Quantization of the Volume Operator according to 2. Method In this subsection we consider the quantization of \mathcal{V} as described in Section 4.3 where the commutator between \hat{p}^I_i and \hat{p}^J_j vanishes. This fact simplifies dramatically the analysis because the (generalized) eigenvalue problem can now be written in terms of products and sums of the eigenfunctions of the \hat{p}^I_i . Let us define

$$T_{\lambda_1, \lambda_{22}, \lambda_{23}, \gamma_{33}, \gamma_{32}} := \mathcal{N}_{\lambda_1} \otimes (f_{\lambda_{22}} g_{\lambda_{23}}) \otimes (f'_{\gamma_{33}} g'_{\gamma_{32}}),$$

where $f_\gamma(Q_2, \theta_1)$, $g_\gamma(Q_2, \theta_1)$, $f'_\gamma(Q_3, \theta_2)$ and $g'_\gamma(Q_3, \theta_2)$ are the (generalized) eigenfunctions of \hat{p}^2_2 , \hat{p}^2_3 , \hat{p}^3_3 and \hat{p}^3_2 respectively given in Section 4.3. Furthermore we denoted the eigenfunctions of \hat{p}^1_1 by $\mathcal{N}_{\lambda_1} := \langle Q_1 | \lambda_1 \rangle$. Since we have

$$(f_{\lambda_{22}} g_{\lambda_{23}})(f'_{\gamma_{33}} g'_{\gamma_{32}}) \propto \frac{\exp(iQ_2(\lambda_{22} \cos \theta_1 - \lambda_{23} \sin \theta_1)) \exp(iQ_3(\gamma_{33} \cos \theta_2 - \gamma_{32} \sin \theta_2))}{\sqrt{\sin \theta_1 \cos \theta_1} \sqrt{\sin \theta_2 \cos \theta_2}}$$

we see that $T_{\lambda_1, \lambda_{22}, \lambda_{23}, \gamma_{33}, \gamma_{32}}$ is not normalizable in \mathcal{H}^S . The generalized eigenvalue problem is thus given by

$$\begin{aligned} \hat{\mathcal{V}} T_{\lambda_1, \lambda_{22}, \lambda_{23}, \gamma_{33}, \gamma_{32}}[\varphi] &= T_{\lambda_1, \lambda_{22}, \lambda_{23}, \gamma_{33}, \gamma_{32}}[\hat{\mathcal{V}}\varphi] \\ &= \gamma^{3/2} l_{\text{Pl}}^3 \sqrt{k} |\lambda_1 (\lambda_{22} \gamma_{33} - \lambda_{23} \gamma_{32})| T_{\lambda_1, \lambda_{22}, \lambda_{23}, \gamma_{33}, \gamma_{32}}[\varphi] \end{aligned} \quad (59)$$

for $\varphi \in \mathcal{D}(\hat{\mathcal{V}})$.

4.5. Quantum Gauss Constraint

In Section 3.4.2 we computed the classical Gauss constraint for a Bianchi type I model. In the open case the elementary variables can always be diagonalized such that both the diffeomorphism and Gauss constraints are automatically satisfied. In the closed model this is not the case anymore so that a quantization of the constraints is mandatory. Since in Bianchi type I models the diffeomorphism constraint is proportional to the Gauss constraint we only need to quantize and solve the latter. However, contrary to the diffeomorphism constraint the Gauss constraint can be quantized infinitesimally.

A gauge transformation of an $su(2)$ -connection is given by

$$A \mapsto A' = \lambda^{-1} A \lambda + \lambda^{-1} d\lambda$$

where $\lambda : \Sigma \mapsto SU(2)$. Infinitesimally we can write this equation as

$$A_a^i \mapsto A_a'^i = A_a^i + \partial_a \epsilon^i + \epsilon^i_{jk} \epsilon^j A_a^k + \mathcal{O}(\epsilon^2).$$

The classical Gauss constraint ensuring $SU(2)$ -invariance is given by

$$G(\Lambda) = - \int_{\mathbb{T}^3} d^3x E_j^a D_a \Lambda^j$$

where $D_a \Lambda^j = \partial_a \Lambda^j + \epsilon^j{}_{kl} A_a^k \Lambda^l$ is the covariant derivative of the smearing field Λ^j . The infinitesimal quantization of this expression yields an operator containing a sum of right and left invariant vector fields over all vertices and edges of a given graph α . This operator is essentially self-adjoint and can, by Stone's theorem, be exponentiated to a unitary operator U_ϕ defining a strongly continuous one-parameter group in ϕ . Usually, in order to find the kernel of the Gauss constraint operator one restrict the scalar product on \mathcal{H}_{aux} to the gauge-invariant scalar product on $\mathcal{H}_{\text{inv}}^G$. This Hilbert space is a true subspace of \mathcal{H}_{aux} since zero is in the discrete part of the spectrum of the Gauss constraint operator.

We saw in Section 3.5 that thanks to the symmetry reduction two of the Gauss constraints are automatically satisfied. While the nonvanishing Gauss constraint (30) is still a complicated function in ϕ_I^i and p^J_j it simplifies to Equation (35) after the canonical transformation. A quantization of this expression is then given by

$$\hat{G}_1 = \hat{P}_{\theta_1} - \hat{P}_{\theta_2}.$$

Since the eigenstates of the momentum operators \hat{P}_{θ_α} are the strict periodic functions satisfying Equation (40) the action of the Gauss constraint on $|\vec{\mu}, \vec{k}\rangle$ is given by

$$\hat{G}_1 |\vec{\mu}, \vec{k}\rangle = \frac{\gamma l_{\text{Pl}}^2}{2} (k_1 - k_2) |\vec{\mu}, \vec{k}\rangle$$

which vanishes if

$$k_1 = k_2.$$

We can thus introduce a new variable $\Theta := \theta_1 + \theta_2$ such that the algebra \mathcal{A}_S given by Equation (38) reduces to the invariant algebra $\mathcal{A}_S^{\text{inv}}$ generated by the functions

$$g(Q_1, Q_2, Q_3, \Theta) = \sum_{\lambda_1, \lambda_2, \lambda_3, k} \xi_{\lambda_1, \lambda_2, \lambda_3, k} \times \exp \left(\frac{1}{2} i \lambda_1 Q_1 + \frac{1}{2} i \lambda_2 Q_2 + \frac{1}{2} i \lambda_3 Q_3 + i k \Theta \right). \quad (60)$$

A Cauchy completion leads to the invariant Hilbert space $\mathcal{H}_{\text{inv}}^S = \mathcal{H}_B^{\otimes 3} \times \mathcal{H}_{S^1}$. A comparison with \mathcal{H}^S shows that we 'lost' one Hilbert space \mathcal{H}_{S^1} by solving the quantum Gauss constraint. Furthermore, instead of two momentum operators conjugated to θ_1 and θ_2 we have just one momentum operator conjugated to Θ defined by

$$\hat{P}_\Theta = -i \gamma l_{\text{Pl}}^2 \frac{\partial}{\partial \Theta}.$$

The eigenstates of all momentum operators are given by

$$|\vec{\mu}, k\rangle := |\mu_1, \mu_2, \mu_3, k\rangle,$$

where $k \in \mathbb{Z}$ defines the representation of $U(1)$.

5. Conclusions and Outlook

In this paper we studied how a torus universe affects the results of LQC. To do so we first introduced the most general tori using Thurston's theorem and found that six Teichmüller parameters are needed. We constructed a metric describing a flat space but respecting the periodicity of the covering group used to construct the torus and used it to derive a gravitational Hamiltonian. We studied the dynamics of a torus universe driven by a homogeneous scalar field by numerically solving the full Hamiltonian and saw that its form only remains cubic if all off-diagonal terms vanish. The Ashtekar connection and the densitized triad for a torus were then derived for both the most general and a slightly simplified torus. The reason for this simplification was that a simple solution to the Gauss constraint could be given. We also derived the Hamiltonian constraint in these new variables and showed that it reduces to the standard constraint of isotropic LQC in case of a cubical torus.

The passage to the quantum theory required a canonical transformation so as to be able to write the holonomies as a product of strictly and almost periodic functions. A Cauchy completion then led to a Hilbert space given by square integrable functions over both \mathbb{R}_B and $U(1)$. However the drawback of the canonical transformation is a much more complicated expression for the components of the densitized triad containing both the momentum and the configuration variables. Following the standard procedure of LQC we substituted these configuration variables with the sine thereof and were able to solve the eigenvalue problem analytically. Surprisingly it turned out that the (generalized) eigenfunctions of the triad operators do not lie in the Hilbert space, i.e. the spectrum is continuous. On the other hand we were also able to find almost periodic solutions to the eigenvalue problem of the triad operators without performing the substitution just described, but once again these eigenfunctions do not lie in the Hilbert space. The reason why both ways lead to a continuous spectrum is the non-cubical form of the torus, for if we set the angles $\theta_{1,2} = 0$ in Equation (32) the triads correspond to the ones obtained in isotropic models. Furthermore we were able to find the spectrum of the volume operator for the second case because, contrary to the first case, it is a product of commuting triad operators.

Although we gave a couple of numerical solutions to the classical Hamiltonian we didn't consider its quantization. The constraint describing quantum dynamics of a torus is given by inserting the holonomies (37) into Thiemann's expression for the quantum Hamiltonian [43]

$$\hat{C}_{\text{grav}} \propto \epsilon^{ijk} \text{tr} \left(h_i^{(0\lambda_i)} h_j^{(0\lambda_j)} (h_i^{(0\lambda_i)})^{-1} (h_j^{(0\lambda_j)})^{-1} h_k^{(0\lambda_k)} [(h_k^{(0\lambda_k)})^{-1}, \hat{V}] \right).$$

Contrary to LQG and LQC we saw that the spectrum of the volume operator of a torus is continuous. It would thus be very interesting to know how far \hat{C}_{grav} departs from the usual difference operator of LQC. Furthermore, whether a quantization of the torus a la LQG removes the Big Bang singularity needs also to be addressed, especially since we saw that many characteristics of both LQG and isotropic LQC are not present in this

particular topology.

In this work we only considered the simplest closed flat topology but there are many other closed topologies. As we saw there are eight geometries admitting at least one compact quotient. For example there are six different compact quotients with covering \mathbb{E}^3 , namely \mathbb{T}^3 , $\mathbb{T}^3/\mathbb{Z}_2$, $\mathbb{T}^3/\mathbb{Z}_3$, $\mathbb{T}^3/\mathbb{Z}_4$, $\mathbb{T}^3/\mathbb{Z}_6$ and a space where all generators are screw motions with rotation angle $\pi/2$. It would be interesting to know how these discrete groups $\mathbb{Z}_{2,3,4,6}$ affect the results of this work, especially since the last five spaces are inhomogeneous (observer dependent) [44].

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Appendix A. Fundamental Domain of the Torus

In two dimensions the upper half-plane H is the set of complex numbers $H = \{x+iy \mid y > 0; x, y \in \mathbb{R}\}$. When endowed with the Poincar e metric $ds^2 = (dx^2 + dy^2)/y^2$ this half-plane is called the Poincar e upper half-plane and is a two-dimensional hyperbolic geometry. The special linear group $SL(2, \mathbb{R})$ acts on H by linear fractional transformations $z \mapsto (az + b)/(cz + d)$, $a, b, c, d \in \mathbb{R}$, and is an isometry group of H since it leaves the Poincar e metric invariant. The modular group $SL(2, \mathbb{Z}) \subset SL(2, \mathbb{R})$ defines a fundamental domain by means of the quotient space $H/SL(2, \mathbb{Z})$. This fundamental domain parametrizes inequivalent families of 2-tori and can thus be identified as the configuration space of the two-dimensional torus. Since we consider a three-dimensional torus with six independent Teichm uller parameters (see Equation (2)) we need a generalization of the Poincar e upper half-plane [45, 46] to a six-dimensional upper half-space.

Definition 6. *A fundamental domain D for $SL(3, \mathbb{Z})$ is a subset of the space $\mathcal{P}_3 := \{A \in Mn(3, \mathbb{R}) \mid A^T = A, A \text{ positive definite}\}$ which is described by the quotient space $\mathcal{P}_3/SL(3, \mathbb{Z})$. In other words, if both $A \in \mathcal{P}_3$ and $A[g] := g^T A g$, $g \in SL(3, \mathbb{Z})$, are in the fundamental domain then either A and $A[g]$ are on the boundary of the fundamental domain or $g = id$.*

Since \mathcal{P}_3 is a subspace of the six-dimensional Euclidean space (there are six independent matrix elements for $A \in \mathcal{P}_3$), the generalization of the Poincar e upper half-plane is now a six-dimensional upper half-space $U^6 := \{(a_1, \dots, a_6) \in \mathbb{E}^6 \mid a_6 > 0\}$ upon which the group $SL(3, \mathbb{R})$ acts transitively. To identify \mathcal{P}_3 with an upper half-space we introduce the Iwasawa coordinates such that $\forall A \in \mathcal{P}_3$ there is the unique decomposition:

$$A = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix},$$

with $x_i, y_j \in \mathbb{R}$ with $\prod y_i = 1$. The geometry of the upper half-space U^6 is given by the $GL(3, \mathbb{R})$ -invariant line element

$$ds^2 = \text{tr}((A^{-1}dA)^2) = \frac{dy_1^2}{y_1^2} + \frac{dy_2^2}{y_2^2} + \frac{dy_3^2}{y_3^2}. \quad (\text{A.1})$$

Note that the Ricci scalar of the metric (A.1) is constant and negative.

In order to give a parametrization of the fundamental region we use Minkowski's reduction theory [47], which tells us that for a metric $h_{i,j}$ the following inequalities must be satisfied:

$$\begin{aligned} h_{i,i} &\leq h_{i+1,i+1}, \quad i = 1, 2, 3 \\ h_{i,j} &\leq \frac{1}{2}h_{i,i}, \quad i < j. \end{aligned}$$

Since the metric (4) is invariant under the map $a_3^3 \rightarrow -a_3^3$ we can define the upper half-space as $U^6 = \{(a_1^1, a_2^1, a_2^2, a_3^1, a_3^2, a_3^3) \in \mathbb{E}^6 \mid a_3^3 > 0\}$, where we have identified the element a_6 with a_3^3 . In our parametrization (4) we therefore obtain the fundamental domain D delimited by the inequalities:

$$\begin{aligned} (a_1^1)^2 &\leq (a_2^1)^2 + (a_2^2)^2 \leq (a_3^1)^2 + (a_3^2)^2 + (a_3^3)^2 \\ a_2^1 &\leq \frac{1}{2}a_1^1 \\ a_3^1 &\leq \frac{1}{2}a_1^1 \\ a_2^1 a_3^1 + a_2^2 a_3^2 &\leq \frac{1}{2}((a_2^1)^2 + (a_2^2)^2) \end{aligned}$$

The first inequality tells us that the length of the generators of the torus are ordered: $\|a_1\| \leq \|a_2\| \leq \|a_3\|$. However, starting with such an ordered triplet does not necessarily imply that the order is preserved by dynamics. Thus we think that it may be more appropriate to choose the equivalent representation of the configuration space given by $\mathcal{C} = \mathbb{R}^6$. Otherwise, we would have to rotate the coordinate system every time the torus leaves the fundamental domain. Note that similar results have also been obtained in M-theory, where one considers a compactification of the extra dimensions on \mathbb{T}^n (see e.g. [48, 49]). However, the situation is different in string theory where one really integrates only over the fundamental domain, e.g. $Z(\mathbb{T}^n) = \int_D d\tau Z(\tau)$.

Appendix B. The Torus Universe in Iwasawa Coordinates

In this appendix we use a parametrization of the torus using the Iwasawa coordinates which are more apt to describe the asymptotic behavior of the metric [50]. It is important to understand the role of the off-diagonal terms in the metric (4) and to know what happens near the singularity and at late times. The metric can be decomposed as follows:

$$h = \mathcal{N}^T \mathcal{D}^2 \mathcal{N}, \quad (\text{B.1})$$

where

$$\mathcal{D} = \begin{pmatrix} e^{-z_1} & 0 & 0 \\ 0 & e^{-z_2} & 0 \\ 0 & 0 & e^{-z_3} \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

An easy calculation shows that Equation (B.1) can be transformed into Equation (4) with $n_1 = a_2^1/a_1^1$, $n_2 = a_3^1/a_1^1$, $n_3 = a_3^2/a_2^2$, $z_i = -\ln a_i^i$ (no summation) \ddagger . The analogue to Equation (3) is now given by

$$h_{ab} = \sum_{i=1}^3 e^{-2z_i} \mathcal{N}_a^i \mathcal{N}_b^i. \quad (\text{B.2})$$

The Iwasawa decomposition can also be viewed as the Gram-Schmidt orthogonalization of the forms dx^a :

$$h_{ab} dx^a \otimes dx^b = \sum_{i=1}^3 e^{-2z_i} \theta^i \otimes \theta^i,$$

where the coframes θ^i are given by

$$\theta^i = \mathcal{N}_a^i dx^a.$$

Analogously, the frames e_i dual to the coframes θ^i are given by the inverse of \mathcal{N}_a^i :

$$e_i = \mathcal{N}^a_i \frac{\partial}{\partial x^a}.$$

Since the determinant of the matrix \mathcal{N} is equal to one the basis given by the coframe is orthonormal. Note that this is different from the construction in Section 2.2.

In order to determine the asymptotic behavior of the off-diagonal terms we follow the analysis in [50]. The metric h being symmetric, we automatically know that its eigenvalues are real. We call these eigenvalues $t^{2\alpha_i}$, $1 \leq i \leq 3$ and $\alpha_1 < \alpha_2 < \alpha_3$, in analogy to the diagonal Kasner solution (see Section 3.4.1) and construct a metric $h_K(t) = \text{diag}(t^{2\alpha_i})$ by means of a constant matrix L

$$h(t) = L^T h_K(t) L, \quad L = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ r_1 & r_2 & r_3 \end{pmatrix}.$$

With these relations we can obtain the evolution of the Iwasawa variables. For example, we have

$$n_1(t) = \frac{t^{2\alpha_1} l_1 l_2 + t^{2\alpha_2} m_1 m_2 + t^{2\alpha_3} r_1 r_2}{t^{2\alpha_1} l_1^2 + t^{2\alpha_2} m_1^2 + t^{2\alpha_3} r_1^2}.$$

In [50] it was shown that the asymptotic behavior $t \rightarrow 0^+$ of the off-diagonal terms of the Iwasawa variables is given by

$$n_1 \rightarrow \frac{l_2}{l_1}, \quad n_2 \rightarrow \frac{l_3}{l_1}, \quad n_3 \rightarrow \frac{l_1 m_3 - l_3 m_1}{l_1 m_2 - l_2 m_1}, \quad (t \rightarrow 0^+),$$

\ddagger For simplicity we assume that all diagonal scale factors a_i^i are strictly positive. However the nondiagonal scale factors can be negative or zero.

which means that the off-diagonal terms freeze in as we approach the singularity. We can also calculate the other limit $t \rightarrow \infty$ and obtain e.g.

$$n_1 \rightarrow \frac{r_2}{r_1}, \quad n_2 \rightarrow \frac{r_3}{r_1}, \quad n_3 \rightarrow \frac{m_1 r_3 - m_3 r_1}{m_1 r_2 - m_2 r_1}, \quad (t \rightarrow \infty).$$

We have checked this result numerically, which can be seen in Figure 2 where the gray line (a_2^{-1}) converges for $t \rightarrow \infty$ toward the solid line (a_1^{-1}), i.e. $n_1 \rightarrow \text{const}$. However, the limit $t \rightarrow 0^+$ could not be checked due to the numerical instability of the solutions when approaching the singularity.

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