Memorandum

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Morphological characterization using the Minkowski Functionals

key-words: integral geometry, n-point correlation functions, image analysis, morphology, topology, Cosmic Microwave Background, statistical data analysis in observational cosmology

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1 Introduction: *Working Group in Morphology*

The elaboration of this memorandum takes place on the background of the exchanges and discussions that I hold with Prof. Dr. Thomas Buchert and Dr. Dr. Claus Beisbart into the field of Mathematical Morphology and its applications to Cosmology. One of the positive realizations, underlying the dynamics of these exchanges is the completion of a preliminary documented investigation upon the subject "*Minkowski functionals to characterize the tilted galaxies morphology*" *[FRA]*.

A role is offered to me to implement and develop a *Working Group in Morphology*, to strengthen and confirm this scientific collaboration, pursuing the tasks and developments endeavored in Statistical Morphology, and supporting interactions with the groups in the GALPAC team and other teams in the laboratory.

With the arrival of Prof. Dr. Frank Steiner and, soon, Dr. Sven Lustig, innovating and strengthening the analysis of observational data, in particular the *CMB* maps from WMAP (and, soon, Planck), I am offered a role to bridge the needs for this scientific collaboration, centered on the high-resolution analysis of *CMB* data (see the envisaged project at the end of this memorandum). This effort has to be viewed in relation to challenging the actual standard model of Cosmology through inhomogeneous cosmological models developed in the Theoretical Cosmology Group.

I will present the foundations of this domain of research which relies upon the mathematical works endeavored by Hermann Minkowski on the geometry of numbers and convex bodies. Over many theoretical progresses made in Mathematics all along the past century, the domain of Integral Geometry ([LAN]) offers now an extremely robust and efficient set of mathematical tools for the objective morphological characterization: the **Minkowski Functionals** (thereafter ***M-F***) .

In Cosmology, the so-called Minkowski Functionals (*M-F*) were developed and adapted first by Mecke K., Buchert T. and Wagner H. in the early nineties to characterize the morphology of large-scale structures of the Universe [MEC-2] in place or in complement of the well-known tool of 2-point autocorrelation functions used up to now. I introduce the *M-F* as an efficient and exhaustive method generalizing (by its concision), correcting (for the degeneracies) and completing (with the Topology) the *k*-point autocorrelation function methods.

I will also summarize the documents written all along the work I made in Galaxy Morphology. It is a preliminary study upon a promising subject of investigations: the "*Robust Morphological Characterization of Tilted Galaxies*"[cf: **Appendix B)**] and *[FRA]*.

2 Mathematical overview

Hermann Minkowski worked on the theory of numbers. His methods of studying quadratic forms and bodies of numbers by geometric means averred as very powerful. Thus, the 'Minkowski theorem' was published in 1891, treated first in 2 dimensions, and generalized later to n dimensions.

Then, the geometry of numbers drives Minkowski to develop novel tools in the geometry of convex bodies (1903) [MIN-1][MIN-2]. *'He introduced a kind of addition of convex sets in three-dimensional*

Euclidean space, and used this addition to associate with any three convex bodies a number which he called their mixed volume. The fundamental importance of these concepts can already be seen from Minkowski's papers and has become particularly apparent in the subsequent presentation of this subject by Bonnesen and Fenchel' (1934) [BON] [GRO].

Therefore, the Minkowski's Sums under affine transformations introduced the concept of mixed volumes for convex sets. These mixed volumes one calls Minkowski Functionals nowadays, a notion introduced in [MEC-2] to replace the germanic word Quermass integrals .

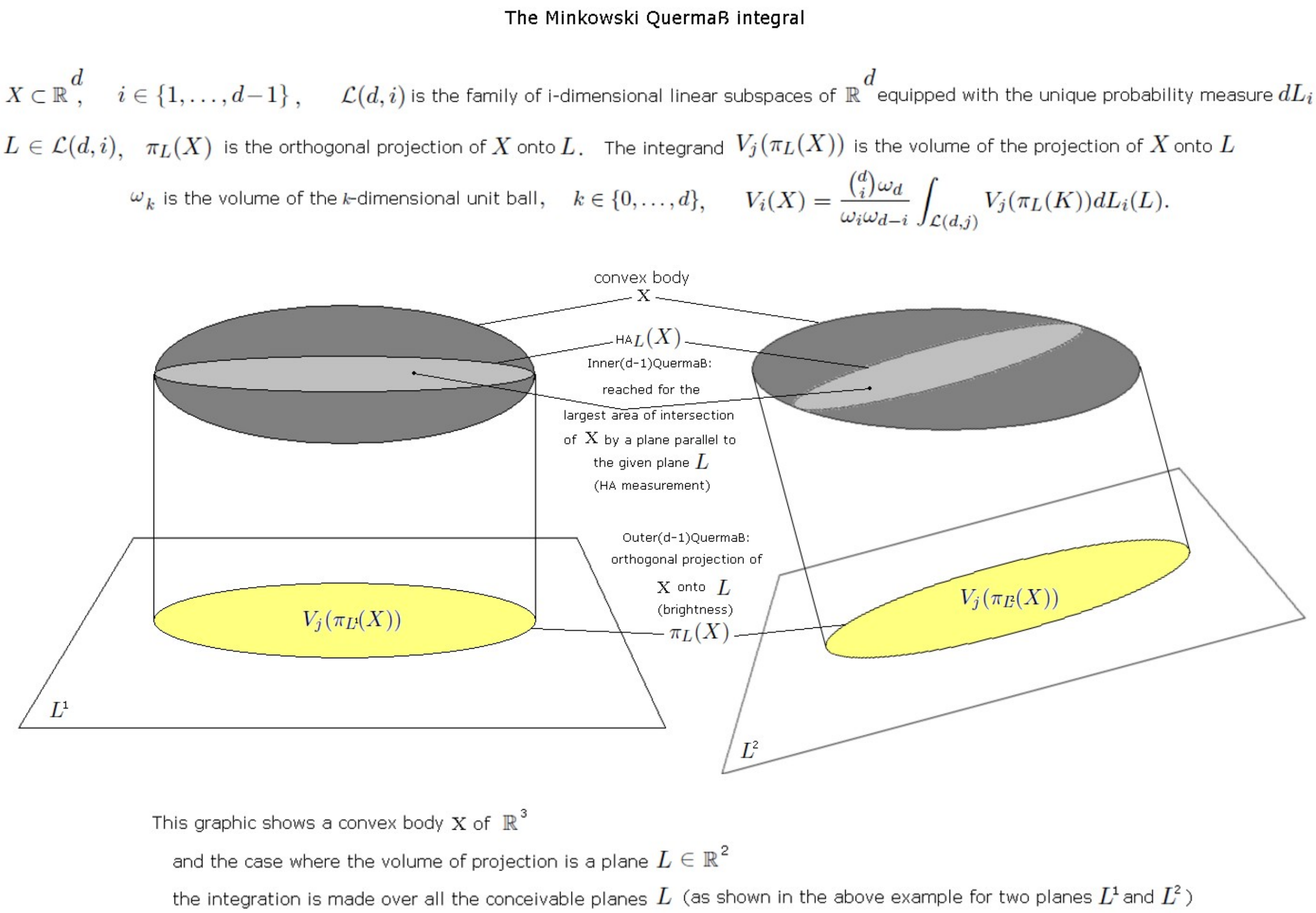


Figure 1 *Principle of the calculation for the Minkowski functionals by the Crofton's Intersection Formula illustrated in the usual 3-D space (graphic by M.J. France)* [TAL][MAR] [SAN][SCHU][HER][AVE]

In 1957, Hugo Hadwiger brings a foundational theorem in stochastic and integral geometry. '*In d spatial dimensions, the global morphological properties (defined as those which satisfy motional invariance, additivity and continuity – see Figure 2) of any pattern can be completely characterized by d+1 numbers, the so-called Quermass integrals'* [SCH-2].

I remind here that, upon a vector space; a scalar- (respectively vector- or tensor-) valued Functional (resp. Valuation) is a function that takes a vector as argument and return a scalar (resp. a vector or a tensor) as output.

'*Any additive, motion invariant and conditionally continuous functional F on subsets* $A \subset E^d, A \in \mathbb{R}$ *is a linear combination of the d+1 Minkowski Functionals*' [MEC-2] (Fig. 2b).

The *n+1* Minkowski Functionals upon a linear vector space *E* of dimension *n* induce a topology on *E* and these *M-F* describe any element of *E* in a unique and complete way.

In 2 and 3 dimensions these *M-F* can be interpreted as intuitive quantities such as volume, surface and the less familiar ones as mean breadth (integrated mean curvature) and connectivity.

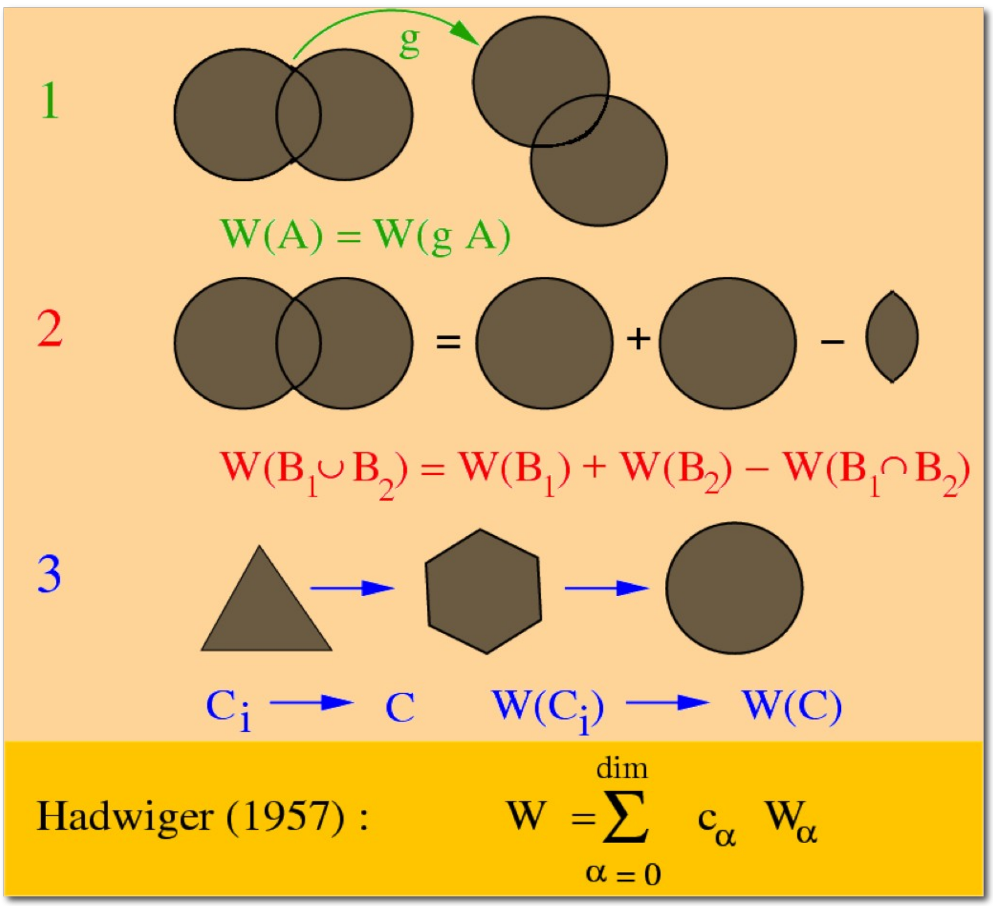


Figure 2 *M-F properties and Hadwiger theorem (graphic by T. Buchert) motion invariance (1) with the group of motions g, additivity (2), and continuity (3)*

3 Cosmological overview

Statistical measures provide important tools for the analysis of the distribution of structures and their intrinsic properties in the Universe. Among numerous statistical methods, *in the cosmological community, the most frequently used measure was and still is the two point correlation function* (autocorrelation) *of a point process such as the distribution of N galaxies* or of larger virialized structures.

Having ρ the volumic density in the sample of galaxies, P_{12} is the probability to have a galaxy centered on the volume dV_1 and another galaxy centered on the volume dV_2.

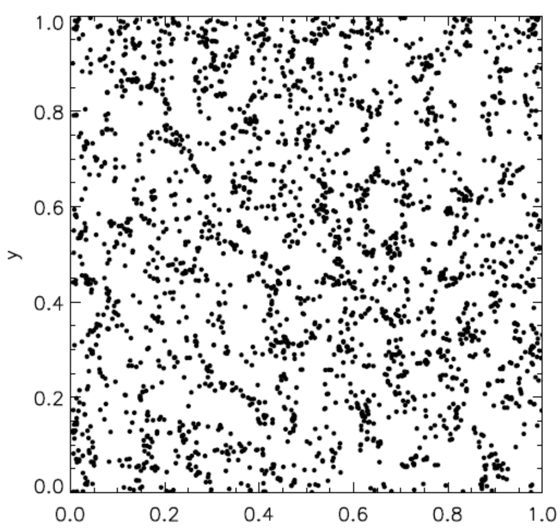
$$P_{12} = \rho^2 (1+\xi_{12})\, dV_1\, dV_2\,,$$

ξ_{12} is the correlation function for two galaxies. In a statistically homogeneous Universe, ξ_{12} seems to depend only on the inter-galaxy distance r. For various galaxy surveys, ξ_{12} adjusts close to

$$\xi_{12} = (r/r_0)^{-1.7}$$

where r_0 = 5 h^{-1} Mpc and for r between 0.5 h^{-1} Mpc and 10 h^{-1} Mpc.
However, on one hand, this estimate is subject to controversies and on the other hand, this does not apply to the larger scales structures (beyond ~ 5 h^{-1} Mpc). Moreover, there are degeneracies; as *completely different spatial patterns could display the same two-point correlation function* [MEC-1].

The following two morphologically different (quite visible on the largest scales) point processes, which have been generated by a Voronoi-model of large-scale structure, show an equal two-point correlation function (by construction) [BUC-1] (Fig. 3).



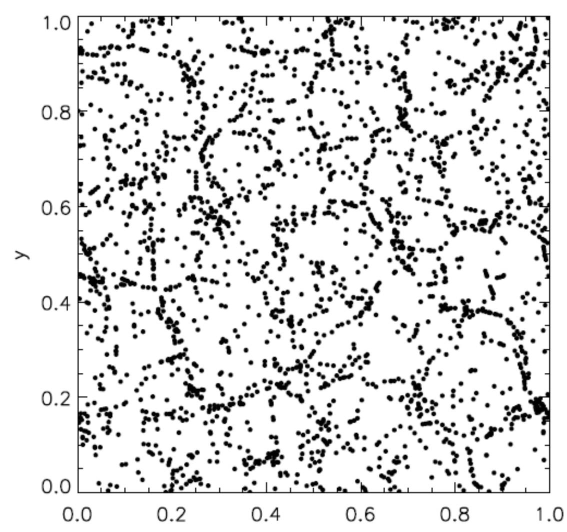


Figure 3 *Two point processes generated by a Voronoi-model (2D projection of ¼ of a 10000 points cube showing identical 2-point autocorrelation functions). T. Buchert.*

The 2-point autocorrelation function is insensitive to the content in phases of a distribution. One can refine the knowledge of a distribution extending the number of points by probability. Thus, we have the autocorrelation function at 3 points defined by the probability P_{123} to have a galaxy centered on the volume dV_1 , another galaxy centered on the volume dV_2 and a third one centered on dV_3:

$$P_{123} = \rho^3\ (1+\xi_{12}+\ \xi_{13}+\ \xi_{23}+\ \xi_{123})\ dV_1\ dV_2\ dV_3\ ,$$

where ξ_{123} is the 3-point autocorrelation function.

Following this way for an increasing number of points, the complete description of our galaxy distribution would be given by the knowledge of all the *k*-point autocorrelation functions (for *k* from 2 to *N ...!*). However, the work, only extended to the third-order (for *k* from 2 to 3) becomes heavy (CPU years or millions of CPU years for a WMAP bispectrum[1]) and complex to implement and interpret.

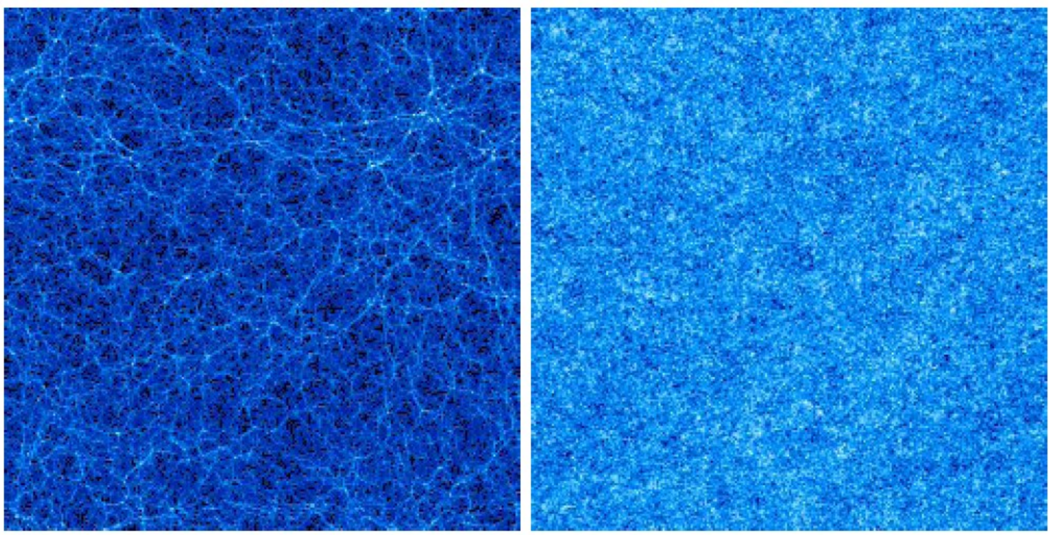


Figure 4 *2 different distributions of Dark Matter matching to similar 2-point correlation functions*

Another cosmological example of two different distributions having similar 2-point correlation functions is taken from I. Szapudi from the University of Hawaii. In Figure 4, the left map shows a mainly filamentary organization (and non-Gaussian distribution) of Cold Dark Matter in a simulation of large-scale structures, while the right map shows a distribution of Cold Dark Matter which is Gaussian. Only the Minkowski Functionals or the higher-rank autocorrelations may differentiate the morphological contents (Fig. 4).

1 These are the Fourier transforms of the k-point autocorrelation functions that are calculated and therefore the notions of power spectrum (for k=2), bispectrum (for k=3) or even trispectrum are used.

4 Robust morphological characterization, generalizing the *k*-point autocorrelations functions

As seen in the theoretical approach (Paragraph 2), the four Minkowski Functionals (*M-F*) in the usual 3-dimensional space offer a concise and complete description of any convex body. Moreover the characterization extends to non-convex bodies using the property of additivity (Fig. 2 property (2)).
Using the discretization method of Klaus Mecke, it was made possible to reach as sharp as needed an accuracy in the morphological analysis. This method adapted the *M-F* to a cloud of as many as required points. One ball of radius R is centered on each point. Thus, these balls can inter-penetrate and R becomes the diagnostic parameter of the point set (Fig. 5).
The three main properties (Additivity, Motion invariance and Conditional Continuity) of *M-F* are kept and, moreover, for instance the small-scale clustering properties of super clusters or galaxy clusters can be known [DRE][FRI][KER-2][KER-4][KER-5][KER-6].

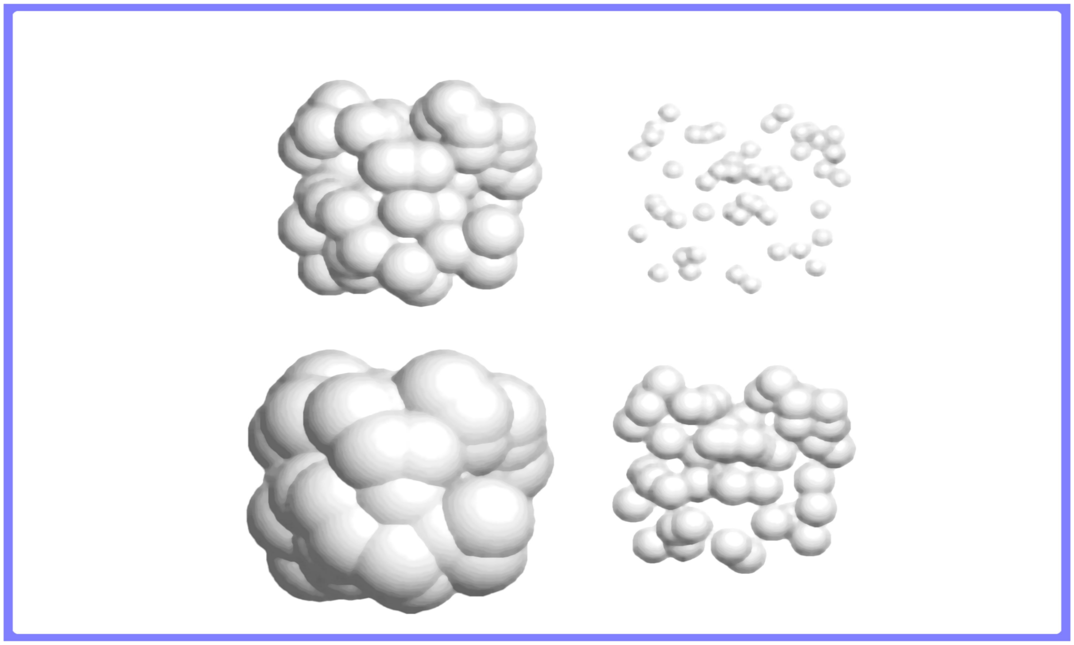


Figure 5 *the boolean grains describing the distribution of galaxies in a cluster*

From the works of Stratonovitch (1963), Mecke and Wagner in 1991 it was proved that the Minkowski Functionals depend on the complete hierarchy of autocorrelations described in Paragraph 3. Morphological Statistics proves that the complex morphological information (of a pattern or a point distribution in a d-dimensional space) contained in the exhaustive set of the k-point autocorrelation functions can be condensed into the concise set of the $d+1$ Minkowski Functionals of Integral Geometry.

Starting from the foundational works made by Mecke K., Buchert T. and Wagner H. [MEC-1] there was an increasing series of developments and numerical applications of the Minkowski Functionals in a domain one may call the Morphological Cosmology. Works and Researches made by Thomas Buchert, Herbert Wagner, Martin Kerscher [KER-1], Jens Schmalzing, Claus Beisbart and David Weinberg (non limiting contributors roster) furbish to Cosmology an evolving set of codes and methods.

5 Computing the Minkowski Functionals

The increasing collection of Minkowski Functionals codes for Cosmology is compiled in the "*Centre de Recherche Astrophysique de Lyon*" (CRAL) and soon made available to the community through a web repository [cf: **Appendix A)**].

The previous international collaboration and the exchanges with the creators and the developers (students, researchers, teams) of these *M-F* codes is maintained and developed for further projects by the Working Group in Morphology, interacting first with the challenges offered by the development of relativistic cosmological models with curvature (related to the Dark Energy and Dark Matter problems [PLA] (http://www.cosmunix.de/arthus/arthus_en.html).

Two main classes of description of the inward data determines the algorithms developed to numerically implement and compute the *M-F*. The CPU-scaling is a main issue.

- *The boolean grains method* describes analytically the discrete 3-D distributions from hundreds (such as the galaxy clusters distribution) up to a few millions of inter-penetrable decorated points [MEC-2] (Fig. 5). For this case, toy-model distributions are easily generated as test-cases.

- *The isodensity contours* method uses the 2-D, 3-D or *n*-D smoothed descriptions of distributions [SCH-6]. The advantage of this method being that any set of data, whatever its size and resolution, can be reliably characterized once the relevant sampling and smoothing is made. This so-called "isocontour" method relies upon the slicing of the meta-image into each of its levels of brightness. For each level the matching binary image can be then processed and the local *M-F* computed for further summation and averaging.
The efficiency of the *M-F* characterization of real world objects remains of course closely dependent on the biases affecting the data. Three biases being more particularly studied in my previous work: the Tilt effect upon the galaxy morphology; the removal of the field exogenous objects and background in the (galaxy) image, and the effects due to the square-cell grid upon the round shaped features.

The practical description of the input dataset can be a meta-data-cube and more specifically a data-grid (such as the usual numerical 2-D images). The yielded output quantities are the *M-F* (which can be scalar, vector or tensor-valued) and the related space of signatures (which put certain morphological features into light). The advanced characterization with the vector- or tensor-valued *M-F* displays its outputs by means of graphics such as the curvature centroids (see Fig. 6 and 7).

6 Research and Applications

The inter-disciplinarity of the Mathematical Morphology using *M-F* is obvious [CAL][JAN][KOE][LEG][MAN]. It is nowadays infrequent to see new advances in pure Mathematics able to offer such an efficient and robust tool as the *M-F* to all the physical sciences. And it is no more frequent to see that one domain, Cosmology, took alone the opportunity to apply, and took a lasting advance to develop this tool with success [MEC-1][PLA][SCH-3][*Applications to data in Cosmology*]. Numerous scientific and technical teams develop and work (sometimes implicitly) with this tool of Integral Geometry, to mention only the fields of Astrophysics [CAN], Biology, Chemistry, Instrumentation and Medicine. Moreover, advances and methods designed for a given research field adapt to another branch. Thus, the 'marching square algorithm using weighted side lengths' I adapted from a numerical implementation dedicated first to chemical analysis [MAN][HO]. The multi-disciplinarity of the *M-F* appears also in the purely mathematical power of the *M-F* in terms of the analysis for any space of phases; thus, chemical strains can be probed whatever the dimension (d) of their space: the d+1 numerical M-F can describe the whole set.

Mathematical developments are slightly unpredictable and we must keep an increasing knowledge and awareness on it. Further algorithmic and numerical developments, Minkowski Valuations (vector-, tensor-valued *M-F*, ...) and derived morphological signatures (filamentarity, planarity, clumpiness,...) retain the highest attention. Is it obvious that the main properties (Fig. 2 (property 1)) of the *M-F* make them insensitive, by definition, to some transformations. Thus, for instance, due to the motion invariance, different relative orientations and positions of the constitutive independent patterns of a subset of $\mathbb{R}^d$ leave the d+1 *M-F* unchanged, while the Minkowski Valuations do the morphological differentiation (Fig. 6) (Fig. 7) [BEI-1].

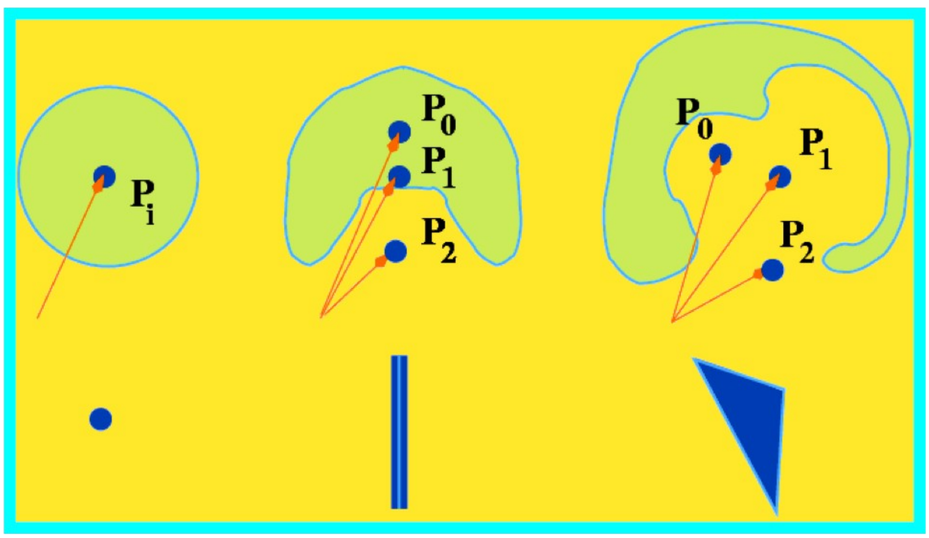


Figure 6 *The Minkowski Valuations (vector-valued) enhance the M-F characterization of the 3 patterns above (graphic by T. Buchert): the Curvature centroids: $\boldsymbol{P_0}$ (for the isobarycentre(s)), $\boldsymbol{P_1}$ (for the circumference(s)) and $\boldsymbol{P_2}$ (for the Euler characteristic(s)).*

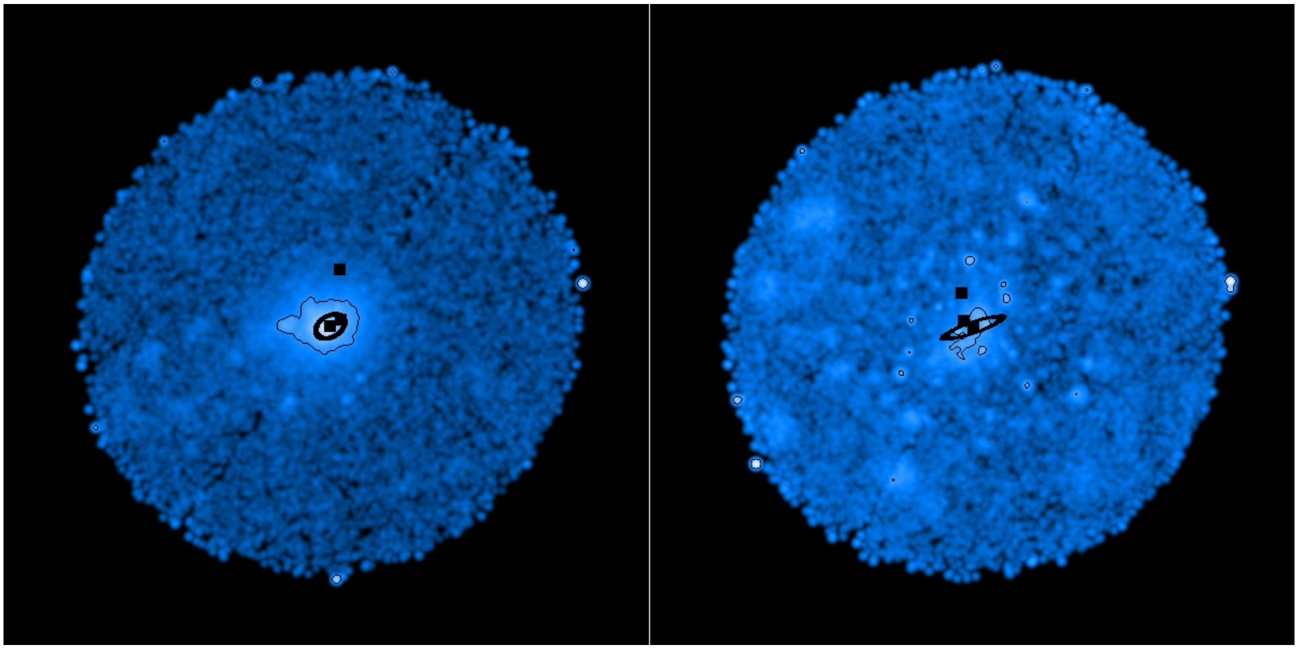


Figure 7 *The Minkowski Valuations (vector- and tensor-valued) applied to galaxy cluster simulations. These 2-D images are some outputs of the code 'mink' by Claus Beisbart. The three square dots are the curvature centroids. The semi-minor and semi-major axis of the central ellipse are proportional to the tensor eigen-values (extrema). This ellipse is oriented along the tensor eigen-direction.*

A Scientific project for the next two years: *Prospect Planck Morphology (P.P.M.)*

With his expertise, Frank Steiner gives us the possibility to analyze the *CMB* data at very high resolution (i.e. 61 500 000 eigen-modes stable map) extending the local curvature knowledge to a global geometric description of the Universe and its various topologies [AUR-x][STE]. Beyond standard methods we expect non-vanishing curvatures and non-trivial topologies. In this context, we devise a statistical new project aimed at the Planck mission.

Motivation of the ***P.P.M.*** project:

Inhomogeneous models predict non-vanishing evolving curvatures (in a void-dominated Universe the averaged curvature is expected to be negative today, while at the *CMB* epoch curvature almost vanishes). Furthermore, F. Steiner and his coworkers showed a closed topological space form to be in better agreement with the actual *CMB* data than the infinite space formed with the standard model. The expectations upon the advanced use of *M-F* for these data and the next to come (*Planck*) are great.

State of the Art:

The Concordance model (Λ-CDM) globally derives from available data in *Observations and Simulations*:

– *Cosmic Microwave Background* (COBE, WMAP, ACT, ...)
– large scale structures (2-DF, COSMOS, SDSS, VVDS, ...; *Virgo consortium, GalICS, Horizon, Mare Nostrum, ...*)
– type Ia supernovae (SDSS, SNLS, SNIFS, ...)

These large datasets are mainly probed with the 2-point autocorrelation functions (***2-pcf***), where even weird results appear as in WMAP: the temperature *2-pcf* at angles greater than 60° are absent. Would an improved Minkowski Functionals analysis confirm this fact?

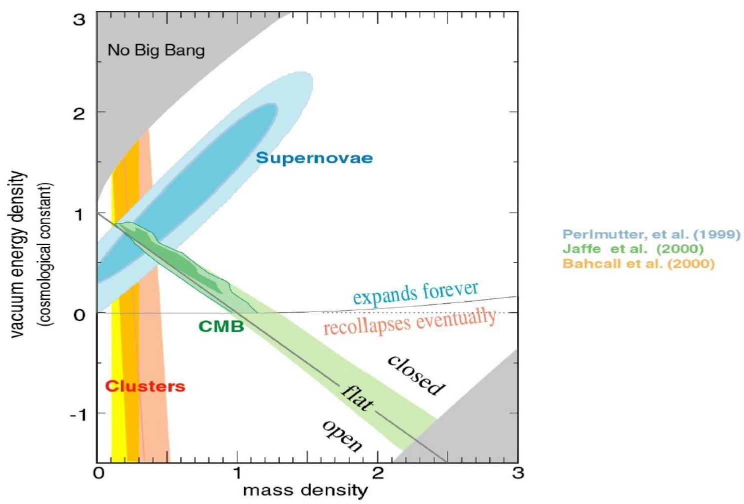


Figure 8 *The keystones of the Concordance model*

The *CMB* is observed through a projection upon the celestial sphere at the unique moment of the photons last scattering. Therefore, one gets rid of the radial component and the study of the *CMB* fluctuations is made using the harmonic functions $f(\theta,\phi)$ *(verifying* $\Delta(f(\theta,\phi))=0, \forall(\theta,\phi)\in E^2$ *)* or ($f(\vec{n})$ where $\vec{n}(\theta,\phi)$ is the direction of the observed photon). On the unitary sphere, one writes $f(\theta,\phi)$ as its harmonic spheric series:

$$f(\vec{n})=\sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l}(a_{lm})(Y_{lm}(\theta,\phi))$$

The *CMB* temperature fluctuations $\frac{\delta T}{T}(\vec{n})$ have the averaged *2-pcf* :

$$C(\vartheta)=\langle\frac{\delta T}{T}(\vec{n_1}(\theta_1,\phi_1))\frac{\delta T}{T}(\vec{n_2}(\theta_2,\phi_2))\rangle \qquad \text{with} \quad \vec{n_1}\circ\vec{n_2}=\cos(\vartheta)$$

Then we have the harmonic spheric series of $C(\vartheta)$:

$$C(\vartheta)=\frac{\text{١}}{\text{٤}\pi}\sum_{l=\text{٠}}^{\infty}(\text{٢}l+\text{١})C_l P_l \cos\vartheta \qquad \text{with} \quad C_l=\frac{1}{2l+1}\sum_{m=-l}^{m=+l}|a_{lm}|^2$$

The infinite series of coefficients C_l characterizes completely the fluctuation spectrum.

Frank Steiner et al. calculated the 61 500 000 first coefficients C_l in a simulation of the *CMB* anisotropy for the Torus (Euclidean geometry) Universe (Fig. 9) [AUR-3].

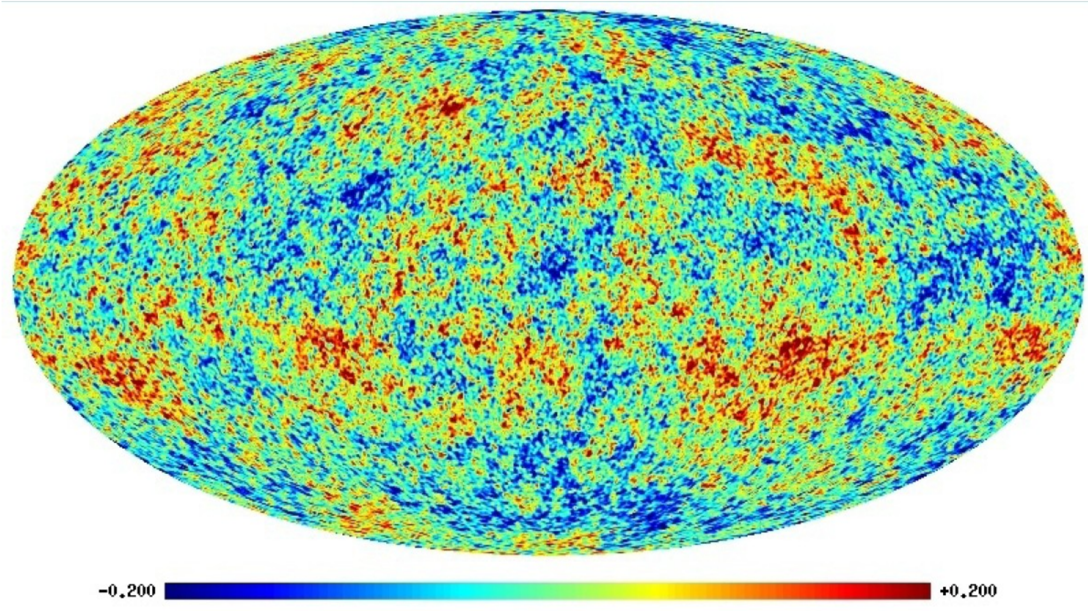


Figure 9 *The CMB Torus Universe simulation (61 500 000 eigenmodes) with L = 4 (Hubble lengths), Aurich, Lustig, Steiner.*

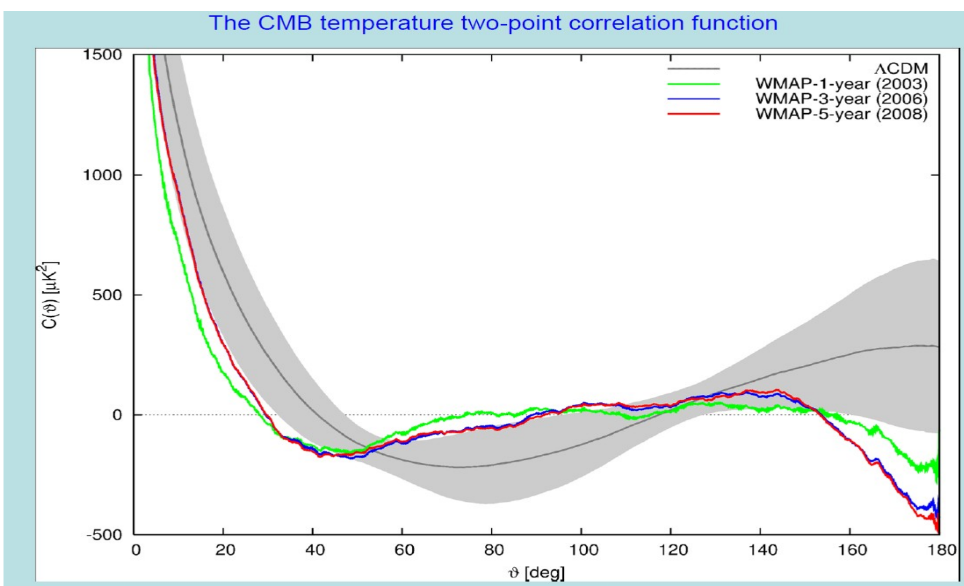


Figure 10 *The WMAP data 2-point autocorrelation function versus the Λ–CDM prediction including cosmic variance (grey area)*

In Figure 10, the 2-pcf curve of the Λ-CDM prediction (and its greyed errors area) does not match to any of the WMAP observations for the autocorrelation angles beyond ~ 60°. WMAP data fit well to a model of a Torus Universe made of finite cells and having a characteristic size, and, therefore imposing a wavelength cut-off and a discrete spectrum [AUR-2].

***CMB* maps and models analyzed with the Minkowski Functionals**:

The COBE DMR data map was scanned with the Minkowski Functionals, and the results proved to be consistent with a Gaussian random field [SCH-4] (1998). Komatsu et al. analyzed the COBE data with the normalized bispectrum approach using the *Healpix* package. They found also the results to be consistent with the Gaussianity of the COBE DMR data [KOM] (2002).
A numerical experiment proposed also by Schmalzing et al. (2000) suggested the study of the weak lensing effect due to large-scale structures upon the *CMB* map. The simulated lens effect induced visible non-Gaussian signatures of the temperature fluctuations through the Minkowski Functionals. Moreover, the numerically simulated experimental specifications of the *Planck* probe let us expect significant anamorphic effects due to weak-lensing upon the *CMB* map [SCH-5].

Hikage et al. analyzed both the WMAP *CMB* and the LSS data, and derived analytical formulas of the *M-F* using a perturbation approach. They put into light the complementarity of the bispectrum (background noise, foreground and mask management) and the scalar-valued *M-F* (systematics in the data) to analyze the primordial non-Gaussianity. Underlining here that, the two methods differ as the *M-F* are intrinsically defined in the real space while the bispectrum is defined in the Fourier space [HIK] (2006).

Matsubara derived the analytical scalar-valued Minkowski Functionals from 2nd order non- Gaussianity including the bispectrum and trispectrum effects in models of primordial density fluctuations. Generating non-Gaussian maps with the *Healpix* package, very good agreements are found with the numerical scalar-valued Minkowski Functionals [MAT] (2010).

Is there any hot pixels contamination in WMAP?

As suggested by Liu et al., an inconsistency between the calibrated Time-Ordered-Data of WMAP and the *CMB* temperature maps from WMAP would exists [LIU]. Therefore, this can put the doubt upon the interpretation of these data for an advanced description of the primordial Universe. The WMAP team refutes the discrepancies revealed by Liu et al. But, a second analysis devised by Aurich, Lustig and Steiner confirmed the bias [AUR-4]. Reconsidering the biases effects due to hot pixels, their team computed the corrected 2-pcf which then displays almost no autocorrelation for the angles greater than 60°. Relying upon that, the above simulation (see Fig. 9) for a Torus Universe remains valid with a slightly smaller cubic fundamental side (L = 4 - ε, measured in units of the Hubble length).

Data Sources:

- The database to analyse current *CMB* maps of WMAP is provided by a NASA web page: (http://lambda.gsfc.nasa.gov/)
- *HEALPix* is now available at: (http://healpix.jpl.nasa.gov). Furthermore, F. Steiner disposes of a code that is considerably faster for the purpose of high-resolution studies.

7 Conclusions and Outlook

In this report, one can underline the importance of the interactions between research in Mathematics and the other fields of investigation; in Cosmology more specifically, about 100 peer-reviewed papers were published upon the Minkowski Functionals and Minkowski Valuations showing their status as a standard tool for advanced applications in Cosmology. We saw that the ever enhanced developments of the theory in Integral Geometry, Hadwiger theorem, Mecke and Wagner contribution and international collaborations, afforded strong improvements and promises to a very accurate knowledge of the distribution and intrinsic properties of structures at any scale in the Universe.

Despite the thorough and independent analysis made by Liu and Aurich et al. concerning the hot pixels of WMAP, a confirmation of the suspected instrumental problem through advanced morphological analysis would be useful. Moreover, Planck should give a better understanding of the foreground (our galaxy) and therefore of the *CMB* sky map. The design or the availability of recent Planck mock maps is expected for preliminary testing on the computer facilities at Ulm and CRAL with the scheduled arrival of Dr. Sven Lustig.

Going one step forward in the timescale; in a collaboration between CRAL (Prof. Dr. T. Buchert) and LAM-Marseille, a student of Dr. Carlo Schimd is currently starting to analyze galaxy catalogues using *M-F* (http://lamwws.oamp.fr/cosmowiki/TopologieLSS). SDSS data will certainly be used for a morphological analysis of the galaxy distribution (some steps beyond the previous investigations) using also the vector-valued *M-F* in collaboration with the Tokyo group [HIK]. The dependencies of the *M-F* on intrinsic magnitude and morphological type of galaxies would be expected to be clearly visible using the Minkowski Valuations. The description of the local and global metrics in a prescription of the inhomogeneous Universe, with big voids challenging the standard Dark Energy interpretation, will benefit of the robustness of the Minkowski Valuations applied to large-scale structures patterns [BUC-2][BUC-3][BUC-4].

The study on the morphology of the individual galaxies (tilted) would also benefit of the enhanced tool using the Minkowski Valuations. To implement this code, a visit of Dr. Dr. Claus Beisbart is planned. The works made on the SAURON survey (kinematical morphology of individual galaxies), and on the ATLAS^{3D} project, will provide data and expertise for the correlation between robust morphology and accurate kinematics. Moreover, thorough works upon the 3-d modelling of the elliptical, ringed and barred-spirals should be the entry point to obtain physically relevant 2-d images of variously tilted galaxies. Some results of the preliminary study would then be confirmed *[FRA]*?

8 Acknowledgments

As a first acknowledgment, dedicated to Steven Weinberg, I must say that the reading in 1981, of 'The first three minutes of the universe', was to me the discovery of both, the high-energy and particules Physics and a novel way to describe and understand the Universe.

I thank Thomas Buchert, Claus Beisbart, Arlette Pécontal, Bruno Guiderdoni and Alban Rémillieux for their constant support and valuable discussions, the constructive criticism and the opportunity they gave me to involve myself in Cosmology.

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Appendix A *The Minkowski Functionals codes*

Boolean grains codes (galaxy clusters distribution, ...):

- ***minkowski1***

'*quer*' by Jens Schmalzing and Martin Kerscher (1995)
A former Program for calculating Quermassintegrals (Minkowski-Functionals) of a 3-D Boolean grain model of simulations
Outer visualization with Gnuplot
cf: [MEC-1], [BUC-1], [PLA], [SCH-3]

- ***minkowski2.x***

'*main*' by Jens Schmalzing and Martin Kerscher (1996, mod. 2007)
A program for calculating Minkowski functionals of a 3-D Boolean grain model with boundary correction
Outer visualization with Gnuplot
cf: [MEC-1], [SCH-3], [KER-2], [CAL]

- ***saha***

'*saha*' by Matthias Ostermann (2007) from the original code by Jens Schmalzing and Andreas Rabus (1998)
SAHA calculates the partial *M-F* of a given point data set by means of the Boolean grain model, then adds them up and determines the global densities (global *M-F* normalized by the density of the point sample)

Isocontours codes:

- ***minkowski3***

'*beyond*' by Jens Schmalzing and Thomas Buchert (1997)
A program for calculating Minkowski functionals of 3-D density fields with periodic boundary conditions
Visualization with SuperMongo
cf: [BUC-1], [KER-3], [KER-2], [MEC-1], [SCH-1], [SCH-2]

- ***minfit***

'*minfit1.1*' by David Weinberg and Martin Kerscher (2000)
A program for calculating Minkowski functionals of 2-D grey-scale images (FITS format used with the cfitsio C library)
Visualizations with .xpm image files
cf: [BEI-1]

- ***mink***

'*mink*' by Claus Beisbart (2003)
A program (structure code and some routines from ***minkowski3***) for calculating 2-D Minkowski valuations (Minkowski vectors and some of the Minkowski tensors) from 3-D point-data or pixel fields (galaxy clusters, ...)
Visualizations with SuperMongo and .xpm image files
cf: [BEI-2], [BEI-3], [SCH-2], [BEI-4]

- ***mfs***

'*mfs*' by Liron Gleser (2005)
A program for calculating Minkowski functionals of 3-D density fields using various methods:
The Crofton's formula, Koenderink invariants; analytical calculation for a Gaussian random field
cf: [GLE-1], [GLE-2], [KOE]

- ***msqw-minfit***

'*msqw-minfit*' by Martin J. France (2009)
A program (developed from the structure code and routines of ***minfit***) for calculating Minkowski functionals of 2-D grey-scale images of galaxies implementing the 'marching square algorithm using weighted side lengths'
Visualizations with GnuPlot and .xpm image files
cf: [MAN-1]

- ***wmap cmb code from former Frank Steiner's student Holger Janzer***

cf. [JAN] – soon available, as well as a comparison code:

cmb code based on 'beyond' from former Thomas Buchert's student Jens Schmalzing

Appendix B

A preliminary investigation: Robust morphological characterization of tilted galaxies

It is of a great interest to test the ranking of a morphologically representative sample of galaxies (such as the samples based upon the 'Hubble' or 'de Vaucouleurs' sequences) applying the Minkowski Functionals characterization to 2-D images of galaxies.

Martin Kerscher (in 2000) worked "*to generate something like an automated morphological classificator for galaxies*" having the SDSS atlas of 2-D images in mind. He wrote a C code (***minfit1.1***) able to read and characterize the FITS images of galaxies with the scalar-valued *M-F*. The method being, for each grey-scale level in the image, to compute the geometric quantities related to the functionals V_0 (surface content), V_1 (length of perimeter) and V_2 (Euler characteristic). Thomas Buchert proposed me to deepen this objective galaxy morphology (as no time was left to M. Kerscher to continue his idea) and to put into prospect the issue of the tilt.

Among the biases one encounters on the way in the analysis of morphological and physical properties of a galaxy through its 2-dimension survey image, one astrophysical bias which is named the "*tilt*" imposes itself as a main issue. Placed in the usual 3-D space, the tilt is the angle made by a plane intrinsic to the given galaxy (such as the plane of biggest observable area) relative to the plane normal to the line of sight. Once the intrinsic plane one refers to is specified, a galaxy showing a tilt angle of 0° is said to be a face-on galaxy and we would like to know the tilt angle of this galaxy by the analysis of its 2-D image. We address and evaluate the questions:

- 'what are the *M-F* results and the related morphological signatures of a sample of galaxy images offering only the natural tilt?'
- 'how the functionals and the morphological signatures evolve with the tilt direction and the tilt angle value of a galaxy type?'
- 'I produced 'tilted' images (using an image processing software) of a given galaxy and tested the stability of the *M-F* and **signatures** (thereafter ***M-F-S***) for the series of tilts'.

An underlying idea of the project being, once obtained a **mock** 'tilted' image of a real galaxy: is the related set of *M-F-S* in good agreement with the *M-F-S* set of a similar real galaxy showing such a **natural** tilt? The a priori Principles of my study are:

- Morphological information resides within each isodensity sub-image of the object.
- The Tilt information can be found back in the *M-F-S*.
- Overall physics (photometry, kinematics, ...) must survive the tilt operation of the 2-D image as galaxies are unchanged whatever the observer situation is.

And the Limits:

- This scalar-valued approach limits the sensitivity due to intrinsic properties of the *M-F*.
- Degree of relevance of the Tilting algorithm based upon image processing assumptions?

In parallel to a bibliographic study (the *M-F* and the galaxy morphology in a cosmological context), my task was to shell the original code of M. Kerscher into a modular application adapting to the further

investigations. This application written in C and mysql, ***msqw-minfit***, I iteratively enriched with an increasing list of features. Features that are essentially; the input modes 'one object image' or 'set of tilted images', the selection of the 'threshold range' or a 'sample of thresholds'. The outputs are numerous, I summarize here: the averaged Minkowski Functionals and signatures by image, some statistical moments and the gnuplot ready datafiles. I implemented the 'Marching Square Algorithm using Weighted Side Lengths' [MAN] into my code to obtain a reliable computing of the functional V_1 related to the geometric quantity C (circumference) of the smoothed patterns.
A summary of the study shows below the roster of nearby galaxies images analyzed with the scalar-valued *M-F*.

A first sample of twelve nearby galaxies and the *M-F* analysis

Galaxies are sorted by rank of the averaged signature **Ms** = $<4\pi S / (\chi C^2)>$ computed with the *msqw-minfit* code over the entire threshold range 38 to 248.

Survey: Digital Sky Survey (DSS1)
Source: Nasa Extragalactic Database (NED) (http://nedwww.ipac.caltech.edu/forms/images.html)
Wavelength: 4680 Å
Band: IIIaJ
Resolution: 1.7 arc sec./pixel

rank	object	Survey filter	Dynamic range (-bit)	Artificial tilt θ (°)	Natural tilt at 4400 angstroëms cos-1(b/a) (°)	Absolute magn. at 4400 A, calc. from B(m_B) and lum. dist.	z	Overall type	de Vaucouleurs classification	M-F signature Ms (at θ =0°)
1	NGC 7020	I IIIaJ dss1	8	0	65	-20.75 +/-0.15	0.010677	ringed spiral	(R)SA(r)0+	0.32330
2	NGC 1374	I IIIaJ dss1	8	0	20	-19.00 +/-0.15	0.004316	elliptical	E3	0.30620
3	NGC 1379	I IIIaJ dss1	8	0	16	-19.08 +/-0.15	0.004416	elliptical	E3	0.24620
4	NGC 1302	I IIIaJ dss1	8	0	29	-20.37 +/-0.15	0.005704	ringed, barred spiral	(R')SAB(rl)a	0.24480
5	NGC 5850	I IIIaJ dss1	8	0	29	-21.09 +/-0.25	0.008526	barred, ringed spiral	SB(r)b	0.19780
6	NGC 1407	I IIIaJ dss1	8	0	21	-21.06 +/-0.15	0.005934	elliptical	E0	0.11720
7	NGC 1640	I IIIaJ dss1	8	0	0	-19.37 +/-0.16	0.005350	barred, ringed spiral	SB(r)b	0.07200
8	NGC 7723 [2]	I IIIaJ dss1	8	0	47	-19.94 +/-0.24	0.006254	barred, ringed spiral	SB(r)b	0.04970
9	NGC 4444	I IIIaJ dss1	8	0	28	-20.60 +/-0.21	0.009737	barred spiral	SAB(rs)bc	0.03740
10	NGC 1300	I IIIaJ dss1	8	0	39	-20.46 +/-0.16	0.005260	barred, ringed spiral	(R')SB(s)bc	0.01002
11	NGC 2835	I IIIaJ dss1	8	0	44	-20.46 +/-0.21	0.002955	barred spiral	SAB(rs)c; HII	0.00214
12	NGC 5247	I IIIaJ dss1	8	0	0	-21.21 +/-0.16	0.004520	spiral	SA(s)bc	0.00131
13	NGC 1365	I IIIaJ dss1	8	0	44	-21.48 +/-0.15	0.005457	barred spiral	(R')SBb(s)b;HIISy1.8	0.00072

Table 1. *Minkowski functionals and signatures of 13 nearby galaxies*

The next pages display the natural survey (DSS1) image and the mock tilted images for two galaxies of the sample. The Minkowski Functionals results are expressed as the averaged quantities of surface <S>, circumference <C> and Euler characteristic <c>.

2 This galaxy is not yet analyzed for tilted mock images

Some results show that:

- As expected from the calculation (derived from the tilting algorithm), the functional <S> contents the tilt angle information (see the bottom right plot for each galaxy).

tilt angle = cos-1(<S>/<S0>)

- The morphological signature *<4 Pi S/(C^2 X)>* remains rather stable whatever the tilt angle values. This morphological stability (tilt invariance) increases with the function (see the upper left and right plots for each galaxy):

$$Ms^{-1} = <4\ Pi\ S/(C\hat{}2\ X)>^{-1}$$

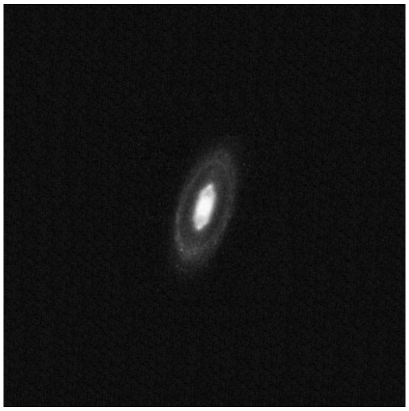
- Back to the untilted images (or natural tilted ones), one can see (Table 1) possible strong correlations between, on one hand, the physical type and overall features of a galaxy and on the other hand, its derived *M-F* signatures such as the above defined **Ms**. I recorded such a behavior among others analyzed samples not displayed herein.

    a) The existence of an outer ring (see for instance the NGC 7020 images below) in a spiral galaxy matches with a strong relative (to the sample) signature Ms.
    b) Ellipticals have also a strong value of Ms.
    c) Spirals having a bar show in the contrary a weak signature Ms (see the last same image below NGC 1365).
    d) Pure spirals show a weak signature Ms.

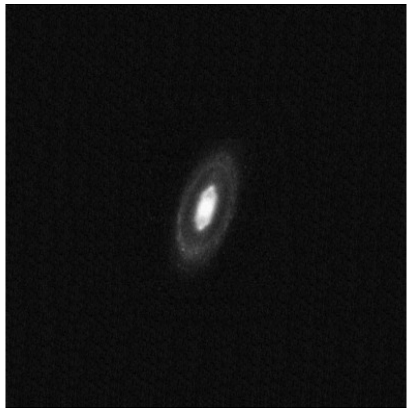
One must keep in mind that the these scalar-valued functionals are motion invariant and therefore, some degeneracies can be encountered as an elliptical may have the same M-F signature than a ringed spiral. Then, the vector- and tensor- valued will discriminate better the morphological features.

The Hubble or the de Vaucouleurs sequences rely upon subjective appreciations and are not physically relevant. On the contrary, it will be useful to compare the Minkowski Functional classifications with other objective classifications such as the one based upon kinematical signatures proposed by Emsellem E. et al. [EMS].

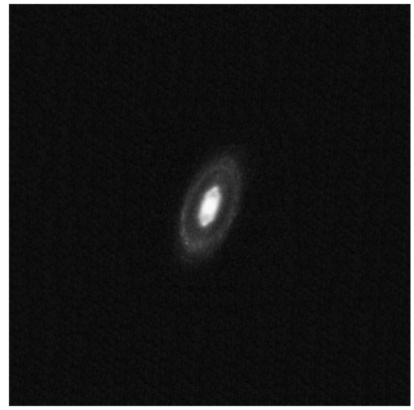
1 NGC 7020 I IIIaJ dss1



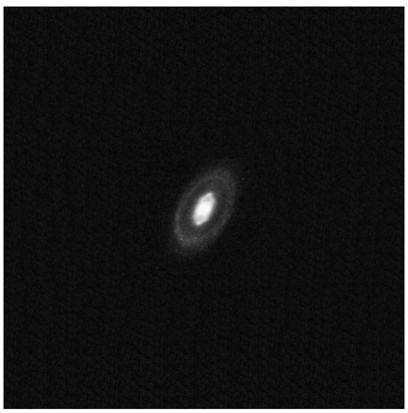
original image, tilt angle = 0.000°



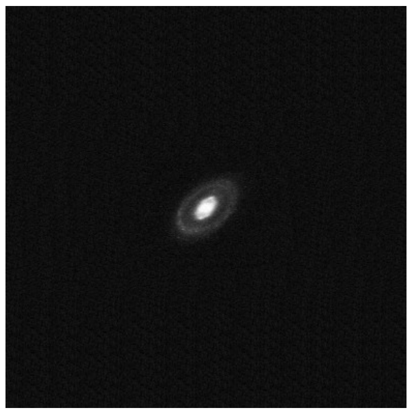
tilt angle = 14.874°



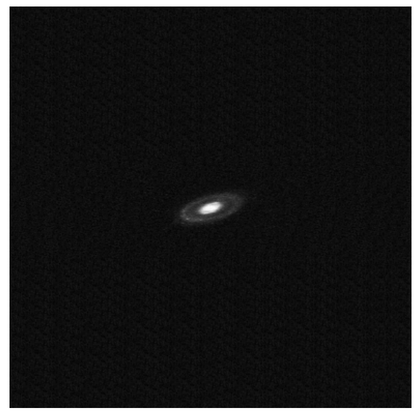
tilt angle = 30.005°



tilt angle = 45.075°

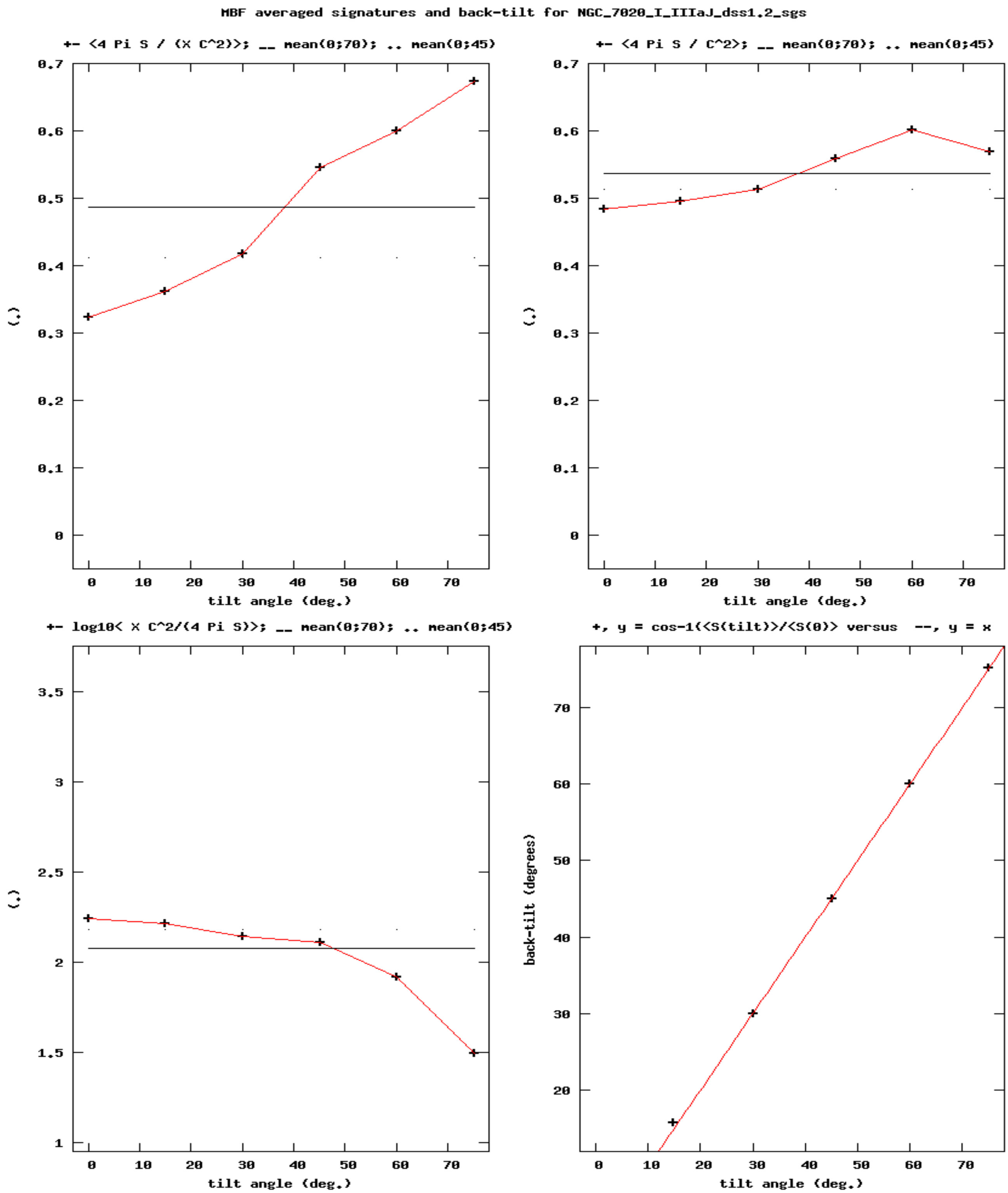


tilt angle = 60.000°

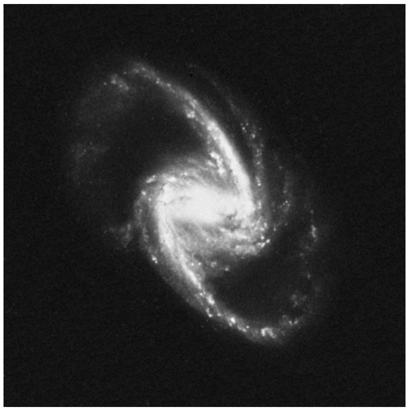


tilt angle = 75.064°

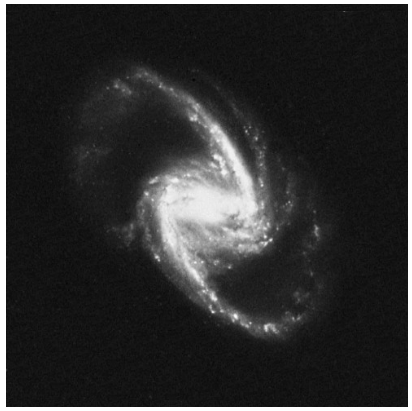
#Axis	angle(°)	<S>	<C>	<chi>	<4*Pi*S/(C^2*X)>	<4*Pi*S/C^2>	<C^2*X/(4*Pi*S)>	cos-1(<S>/<S0>)
1	0.000	0.0055	0.5751	8	0.3233	0.4840	174.6645	
1	14.874	0.0053	0.5577	7	0.3613	0.4956	163.0462	15.788
1	30.005	0.0048	0.5134	7	0.4178	0.5136	139.1066	29.967
1	45.075	0.0039	0.4412	7	0.5452	0.5583	128.2997	45.025
1	60.000	0.0027	0.3427	6	0.6007	0.6012	82.1060	60.065
1	75.064	0.0014	0.2237	4	0.6730	0.5688	31.0975	75.114
#					--> Averaging per axis:			
1					0.4869	0.5369	119.7201	
#					--> Averaging per axis; angle range 0° to 45°:			
#1					0.4119	0.5129	151.2793	


MBF averaged signatures and back-tilt for NGC_7020_I_IIIaJ_dss1.2_sgs
+- <4 Pi S / (X C^2)>; __ mean(0;70); .. mean(0;45)
+- <4 Pi S / C^2>; __ mean(0;70); .. mean(0;45)
+- log10< X C^2/(4 Pi S)>; __ mean(0;70); .. mean(0;45)
+, y = cos-1(<S(tilt)>/<S(0)> versus --, y = x
(.)
back-tilt (degrees)
tilt angle (deg.)

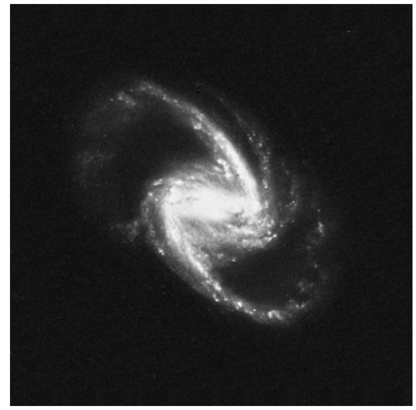
12 NGC 1365 I IIIaJ dss1



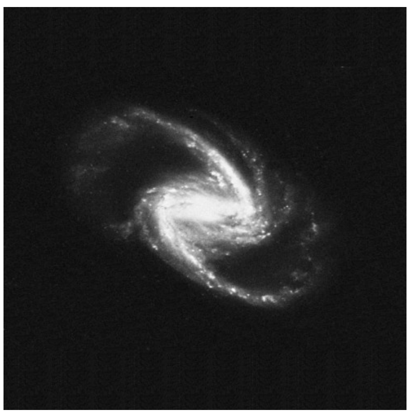
original image, tilt angle = 0.000°



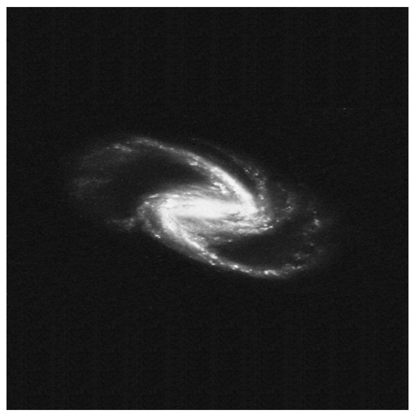
tilt angle = 15.075°



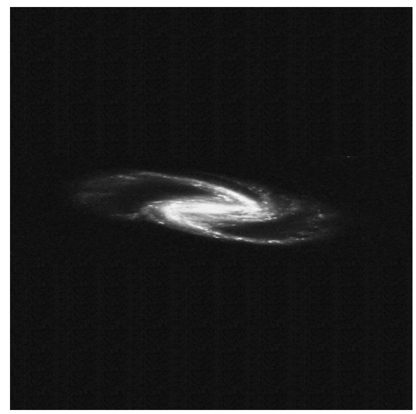
tilt angle = 29.957°



tilt angle = 45.051°

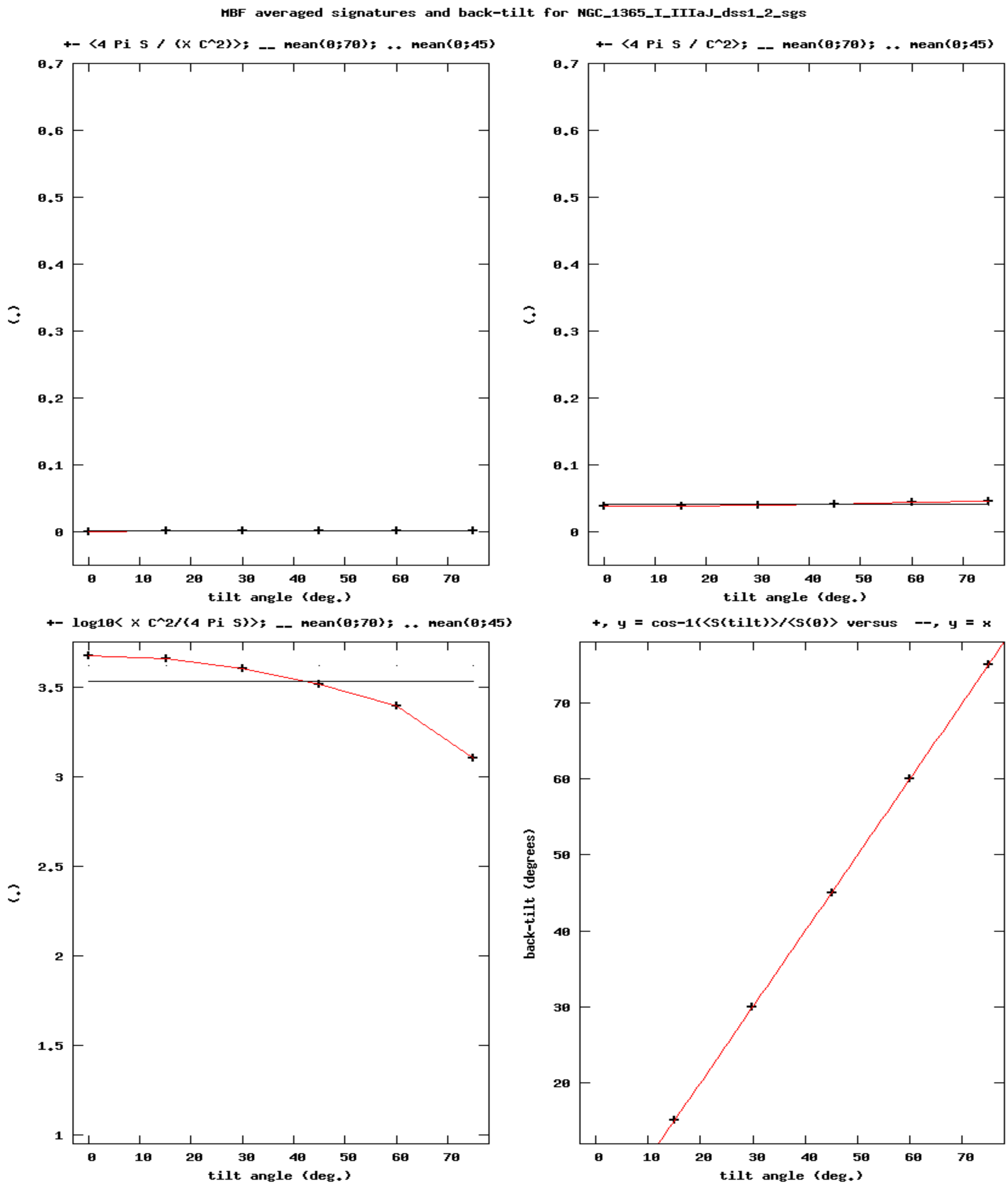


tilt angle = 60.000°



tilt angle = 74.983°

#Axis	angle(°)	<S>	<C>	<chi>	<4*Pi*S/(C^2*X)>	<4*Pi*S/C^2>	<C^2*X/(4*Pi*S)>	cos-1(<S>/<S0>)
1	0.000	0.0677	4.7287	89	0.00072	0.0377	4720.7696	
1	15.075	0.0653	4.6159	88	0.00074	0.0382	4546.875	15.077
1	29.957	0.0586	4.2694	83	0.00081	0.0398	4021.9691	30.025
1	45.051	0.0478	3.7360	77	0.00088	0.0418	3266.5453	45.039
1	60.000	0.0338	3.0506	67	0.00104	0.0437	2475.2907	60.017
1	74.983	0.0175	2.1250	42	0.00215	0.0449	1260.5644	74.990
#					--> Averaging per axis:			
1					0.0010	0.0410	3382.0023	
#					--> Averaging per axis; angle range 0° to 45°:			
#1					0.0007	0.0394	4139.0397	


MBF averaged signatures and back-tilt for NGC_1365_I_IIIaJ_dss1_2_sgs
+- <4 Pi S / (X C^2)>; __ mean(0;70); .. mean(0;45)
+- <4 Pi S / C^2>; __ mean(0;70); .. mean(0;45)
+- log10< X C^2/(4 Pi S)>; __ mean(0;70); .. mean(0;45)
+, y = cos-1(<S(tilt)>/<S(0)> versus --, y = x
(.)
back-tilt (degrees)
tilt angle (deg.)